

SECTORIAL ANALYTIC NORMALIZATION FOR A CLASS OF DOUBLY-RESONANT SADDLE-NODE VECTOR FIELDS IN $(\mathbb{C}^3, 0)$

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ABSTRACT. In this work, following [Bit16], we consider germs of analytic singular vector fields in \mathbb{C}^3 with an isolated and doubly-resonant singularity of saddle-node type at the origin. Such vector fields come from irregular two-dimensional differential systems with two opposite non-zero eigenvalues, and appear for instance when studying the irregular singularity at infinity in Painlevé equations $(P_j)_{j=1, \dots, V}$ for generic values of the parameters. Under suitable assumptions, we prove a theorem of analytic normalization over sectorial domains, analogous to the classical one due to Hukuhara-Kimura-Matuda [HKM61] for saddle-nodes in \mathbb{C}^2 . We also prove that the normalizing map is essentially unique and *weakly Gevrey-1 summable*.

1. INTRODUCTION

As in [Bit16], we consider (germs of) singular vector fields Y in \mathbb{C}^3 which can be written in appropriate coordinates $(x, \mathbf{y}) := (x, y_1, y_2)$ as

$$(1.1) \quad Y = x^2 \frac{\partial}{\partial x} + \left(-\lambda y_1 + F_1(x, \mathbf{y}) \right) \frac{\partial}{\partial y_1} + \left(\lambda y_2 + F_2(x, \mathbf{y}) \right) \frac{\partial}{\partial y_2} ,$$

where $\lambda \in \mathbb{C}^*$ and F_1, F_2 are germs of holomorphic functions in $(\mathbb{C}^3, 0)$ of homogeneous valuation (order) at least two. They represent irregular two-dimensional differential systems having two opposite non-zero eigenvalues:

$$\begin{cases} x^2 \frac{dy_1(x)}{dx} = -\lambda y_1(x) + F_1(x, \mathbf{y}(x)) \\ x^2 \frac{dy_2(x)}{dx} = \lambda y_2(x) + F_2(x, \mathbf{y}(x)) \end{cases} .$$

These we call doubly-resonant vector fields of saddle-node type (or simply **doubly-resonant saddle-nodes**). We will impose more (non-generic) conditions in the sequel. The motivation for studying such vector fields is at least of two types.

- (1) There are two independent resonance relations between the eigenvalues (here $0, -\lambda$ and λ): we generalize then the study in [MR82, MR83]. More generally, this work is aimed at understanding singularities of vector fields in \mathbb{C}^3 . According to a theorem of resolution of singularities in dimension less than three in [MP⁺13], there exists a list of “final models” for singularities (*log-canonical*) obtained after a finite procedure of *weighted blow-ups* for three dimensional singular analytic vector fields. In this list, we find in particular doubly-resonant saddles-nodes, as those we are interested in. In dimension 2, these final models have been intensively studied (for instance by Martinet, Ramis, Ecalle, Ilyashenko, Teyssier, ...) from the view point of both formal and analytic classification (some important questions remain unsolved, though). In dimension 3, the problems of formal and analytic classification are still open questions, although Stolovitch has performed such a classification for 1-resonant vector fields in any dimension [Sto96]. The presence of two kinds of resonance relations brings new difficulties.

Key words and phrases. Painlevé equations, singular vector field, irregular singularity, resonant singularity, Gevrey-1 summability.

- (2) Our second main motivation is the study of the irregular singularity at infinity in Painlevé equations $(P_j)_{j=I,\dots,V}$, for generic values of the parameters (*cf.* [Yos85]). These equations were discovered by Paul Painlevé [Pai02] because the only movable singularities of the solutions are poles (the so-called *Painlevé property*). Their study has become a rich domain of research since the important work of Okamoto [Oka77]. The fixed singularities of the Painlevé equations, and more particularly those at infinity, were notably investigated by Boutroux with his famous *tritrinquées* solutions [Bou13]. Recently, several authors provided more complete information about such singularities, studying “quasi-linear Stokes phenomena” and also giving connection formulas; we refer to the following (non-exhaustive) sources [JK92, Kap04, KK93, JK01, CM82, CCH15]. Stokes coefficients are invariant under local changes of analytic coordinates, but do not form a complete invariant of the vector field. To the best of our knowledge there currently does not exist a general analytic classification for doubly-resonant saddle-nodes. Such a classification would provide a new framework allowing to analyze Stokes phenomena in that class of singularities.

In this paper we provide a theorem of analytic normalization over sectorial domain (*à la* Hukuhara-Kimura-Matuda [HKM61] for saddle-nodes in $(\mathbb{C}^2, 0)$) for a specific class (to be defined later on) of doubly-resonant saddle-nodes which contains the Painlevé case. In a forthcoming paper we use this theorem in order to provide a complete analytic classification for this class of vector fields, based on the ideas in the important works [MR82, MR83, Sto96].

In [Yos84, Yos85] Yoshida shows that doubly-resonant saddle-nodes arising from the compactification of Painlevé equations $(P_j)_{j=I,\dots,V}$ (for generic values for the parameters) are conjugate to vector fields of the form:

$$(1.2) \quad \begin{aligned} Z = & \quad x^2 \frac{\partial}{\partial x} + \left(-(1 + \gamma y_1 y_2) + a_1 x \right) y_1 \frac{\partial}{\partial y_1} \\ & + \left(1 + \gamma y_1 y_2 + a_2 x \right) y_2 \frac{\partial}{\partial y_2} \quad , \end{aligned}$$

with $\gamma \in \mathbb{C}^*$ and $(a_1, a_2) \in \mathbb{C}^2$ such that $a_1 + a_2 = 1$. One should notice straight away that this “conjugacy” does not agree with what is traditionally (in particular in this paper) meant by conjugacy, for Yoshida’s transform $\Psi(x, \mathbf{y}) = (x, \psi_1(x, \mathbf{y}), \psi_2(x, \mathbf{y}))$ takes the form

$$(1.3) \quad \psi_i(x, \mathbf{y}) = y_i \left(1 + \sum_{\substack{(k_0, k_1, k_2) \in \mathbb{N}^3 \\ k_1 + k_2 \geq 1}} \frac{q_{i, \mathbf{k}}(x)}{x^{k_0}} y_1^{k_1 + k_0} y_2^{k_1 + k_0} \right) \quad ,$$

where each $q_{i, \mathbf{k}}$ is formal power series although x appears with negative exponents. This expansion may not even be a formal Laurent series. It is, though, the asymptotic expansion along $\{x = 0\}$ of a function analytic in a domain

$$\left\{ (x, \mathbf{z}) \in S \times \mathbf{D}(0, \mathbf{r}) \mid |z_1 z_2| < \nu |x| \right\}$$

for some small $\nu > 0$, where S is a sector of opening greater than π with vertex at the origin and $\mathbf{D}(0, \mathbf{r})$ is a polydisc of small poly-radius $\mathbf{r} = (r_1, r_2) \in (\mathbb{R}_{>0})^2$. Moreover the $(q_{i, \mathbf{k}}(x))_{i, \mathbf{k}}$ are actually Gevrey-1 power series. The drawback here is that the transforms are convergent on regions so small that taken together they cannot cover an entire neighborhood of the origin in \mathbb{C}^3 (which seems to be problematic to obtain an analytic classification *à la* Martinet-Ramis).

Several authors studied the problem of convergence of formal transformations putting vector fields as in (1.1) into “normal forms”. Shimomura, improving on a result of Iwano [Iwa80], shows in [Shi83] that analytic doubly-resonant saddle-nodes satisfying more restrictive conditions are conjugate (formally

and over sectors) to vector fields of the form

$$x^2 \frac{\partial}{\partial x} + (-\lambda + a_1 x) y_1 \frac{\partial}{\partial y_1} + (\lambda + a_2 x) y_2 \frac{\partial}{\partial y_2}$$

via a diffeomorphism whose coefficients have asymptotic expansions as $x \rightarrow 0$ in sectors of opening greater than π .

Stolovitch then generalized this result to any dimension in [Sto96]. More precisely, Stolovitch's work offers an analytic classification of vector fields in \mathbb{C}^{n+1} with an irregular singular point, without further hypothesis on eventual additional resonance relations between eigenvalues of the linear part. However, as Iwano and Shimomura did, he needed to impose other assumptions, among which the condition that the restriction of the vector field to the invariant hypersurface $\{x = 0\}$ is a linear vector field. In [BDM08], the authors obtain a *Gevrey-1 summable* “normal form”, though not as simple as Stolovitch's one and not unique *a priori*, but for more general kind of vector field with one zero eigenvalue. However, the same assumption on hypersurface $\{x = 0\}$ is required (the restriction is a linear vector field). Yet from [Yos85] stems the fact that this condition is not met in the case of Painlevé equations $(P_j)_{j=I, \dots, V}$.

In comparison, we merely ask here that the restricted vector field be orbitally linearizable (see Definition 1.7), *i.e.* the foliation induced by Y on $\{x = 0\}$ (and not the vector field $Y|_{\{x=0\}}$ itself) be linearizable. The fact that this condition is fulfilled by the singularities of Painlevé equations formerly described is well-known.

1.1. Scope of the paper.

The action of local analytic / formal diffeomorphisms Ψ fixing the origin on local holomorphic vector fields Y of type (1.1) by change of coordinates is given by

$$\Psi_* Y := D\Psi(Y) \circ \Psi^{-1}.$$

In [Bit16] we performed the formal classification of such vector fields by exhibiting an explicit universal family of vector fields for the action of formal changes of coordinates at 0 (called a family of normal forms). Such a result seems currently out of reach in the analytic category: it is unlikely that an explicit universal family for the action of local analytic changes of coordinates be described anytime soon. If we want to describe the space of equivalent classes (of germs of a doubly-resonant saddle-node under local analytic changes of coordinates) with same formal normal form, we therefore need to find a complete set of invariants which is of a different nature. We call **moduli space** this quotient space and would like to give it a (non-trivial) presentation based on functional invariants *à la* Martinet-Ramis [MR82, MR83].

The main ingredient to obtain such analytic invariant is to prove first the existence of analytic sectorial normalizing maps (over a pair of opposite “wide” sectors of opening greater than π whose union covers a full punctured neighborhood of $\{x = 0\}$). This is the main result of the present paper. We have not been able to perform this normalization in such a generality, and only deal here with x -fibered local analytic conjugacies acting on vector fields of the form (1.1) with some additional assumptions detailed further down (see Definitions 1.1, 1.3 and 1.7). Importantly, these hypothesis are met in the case of Painlevé equations mentioned above.

Our approach has some geometric flavor, since we avoid the use of fixed-point methods altogether to establish the existence of the normalizing maps, and generalize instead the approach of Teyssier [Tey04, Tey03] relying on path-integration of well-chosen 1-forms (following Arnold's method of characteristics [Arn74]).

As a by-product of this normalization we deduce that the normalizing sectorial diffeomorphisms are *weakly* Gevrey-1 asymptotic to the normalizing formal power series of [Bit16], retrospectively proving their *weak 1-summability* (see subsection 2.3 for definition). When the vector field additionally supports a symplectic transverse structure (which is again the case of Painlevé equations) we prove that the (essentially unique) sectorial normalizing map is realized by a transversally symplectic diffeomorphism.

1.2. Definitions and main results.

To state our main results we need to introduce some notations and nomenclature.

- For $n \in \mathbb{N}_{>0}$, we denote by $(\mathbb{C}^n, 0)$ an (arbitrary small) open neighborhood of the origin in \mathbb{C}^n .
- We denote by $\mathbb{C}\{x, \mathbf{y}\}$, with $\mathbf{y} = (y_1, y_2)$, the \mathbb{C} -algebra of germs of holomorphic functions at the origin of \mathbb{C}^3 , and by $\mathbb{C}\{x, \mathbf{y}\}^\times$ the group of invertible elements for the multiplication (also called units), *i.e.* elements U such that $U(0) \neq 0$.
- $\chi(\mathbb{C}^3, 0)$ is the Lie algebra of germs of singular holomorphic vector fields at the origin \mathbb{C}^3 . Any vector field in $\chi(\mathbb{C}^3, 0)$ can be written as

$$Y = b(x, y_1, y_2) \frac{\partial}{\partial x} + b_1(x, y_1, y_2) \frac{\partial}{\partial y_1} + b_2(x, y_1, y_2) \frac{\partial}{\partial y_2}$$

with $b, b_1, b_2 \in \mathbb{C}\{x, y_1, y_2\}$ vanishing at the origin.

- $\text{Diff}(\mathbb{C}^3, 0)$ is the group of germs of a holomorphic diffeomorphism fixing the origin of \mathbb{C}^3 . It acts on $\chi(\mathbb{C}^3, 0)$ by conjugacy: for all

$$(\Phi, Y) \in \text{Diff}(\mathbb{C}^3, 0) \times \chi(\mathbb{C}^3, 0)$$

we define the push-forward of Y by Φ by

$$(1.4) \quad \Phi_*(Y) := (D\Phi \cdot Y) \circ \Phi^{-1} \quad ,$$

where $D\Phi$ is the Jacobian matrix of Φ .

- $\text{Diff}_{\text{fib}}(\mathbb{C}^3, 0)$ is the subgroup of $\text{Diff}(\mathbb{C}^3, 0)$ of fibered diffeomorphisms preserving the x -coordinate, *i.e.* of the form $(x, \mathbf{y}) \mapsto (x, \phi(x, \mathbf{y}))$.
- We denote by $\text{Diff}_{\text{fib}}(\mathbb{C}^3, 0; \text{Id})$ the subgroup of $\text{Diff}_{\text{fib}}(\mathbb{C}^3, 0)$ formed by diffeomorphisms tangent to the identity.

All these concepts have *formal* analogues, where we only suppose that the objects are defined with formal power series, not necessarily convergent near the origin.

Definition 1.1. A **diagonal doubly-resonant saddle-node** is a vector field $Y \in \chi(\mathbb{C}^3, 0)$ of the form

$$Y = x^2 \frac{\partial}{\partial x} + \left(-\lambda y_1 + F_1(x, \mathbf{y}) \right) \frac{\partial}{\partial y_1} + \left(\lambda y_2 + F_2(x, \mathbf{y}) \right) \frac{\partial}{\partial y_2} \quad ,$$

with $\lambda \in \mathbb{C}^*$ and $F_1, F_2 \in \mathbb{C}\{x, \mathbf{y}\}$ of order at least two. We denote by $\mathcal{SN}_{\text{diag}}$ the set of such vector fields.

Remark 1.2. One can also define the foliation associate to a diagonal doubly-resonant saddle-node in a geometric way. A vector field $Y \in \chi(\mathbb{C}^3, 0)$ is orbitally equivalent to a diagonal doubly-resonant saddle-node (*i.e.* Y is conjugate to some VX , where $V \in \mathbb{C}\{x, \mathbf{y}\}^\times$ and $X \in \mathcal{SN}_{\text{fib}}$) if and only if the following conditions hold:

- (1) $\text{Spec}(D_0 Y) = \{0, -\lambda, \lambda\}$ with $\lambda \neq 0$;
- (2) there exists a germ of irreducible analytic hypersurface $\mathcal{H}_0 = \{S = 0\}$ which is transverse to the eigenspace E_0 (corresponding to the zero eigenvalue) at the origin, and which is stable under the flow of Y ;
- (3) $\mathcal{L}_Y(S) = U.S^2$, where \mathcal{L}_Y is the Lie derivative of Y and $U \in \mathbb{C}\{x, \mathbf{y}\}^\times$.

By Taylor expansion up to order 1 with respect to \mathbf{y} , given a vector field $Y \in \mathcal{SN}_{\text{diag}}$ written as in (1.1) we can consider the associate 2-dimensional system:

$$(1.5) \quad x^2 \frac{d\mathbf{y}}{dx} = \alpha(x) + \mathbf{A}(x) \mathbf{y}(x) + \mathbf{F}(x, \mathbf{y}(x)) \quad ,$$

with $\mathbf{y} = (y_1, y_2)$, such that the following conditions hold:

- $\alpha(x) = \begin{pmatrix} \alpha_1(x) \\ \alpha_2(x) \end{pmatrix}$, with $\alpha_1, \alpha_2 \in \mathbb{C}\{x\}$ and $\alpha_1, \alpha_2 \in \mathcal{O}(x^2)$
- $\mathbf{A}(x) \in \text{Mat}_{2,2}(\mathbb{C}\{x\})$ with $\mathbf{A}(0) = \text{diag}(-\lambda, \lambda)$, $\lambda \in \mathbb{C}^*$

- $\mathbf{F}(x, \mathbf{y}) = \begin{pmatrix} F_1(x, \mathbf{y}) \\ F_2(x, \mathbf{y}) \end{pmatrix}$, with $F_1, F_2 \in \mathbb{C}\{x, \mathbf{y}\}$ and $F_1, F_2 \in \mathcal{O}(\|\mathbf{y}\|^2)$.

Based on this expression, we state:

Definition 1.3. The **residue** of $Y \in \mathcal{SN}_{\text{diag}}$ is the complex number

$$\text{res}(Y) := \left(\frac{\text{Tr}(\mathbf{A}(x))}{x} \right)_{|x=0}.$$

We say that Y is **non-degenerate** (*resp.* **strictly non-degenerate**) if $\text{res}(Y) \notin \mathbb{Q}_{\leq 0}$ (*resp.* $\Re(\text{res}(Y)) > 0$).

Remark 1.4. It is obvious that there is an action of $\text{Diff}_{\text{fib}}(\mathbb{C}^3, 0, \text{Id})$ on $\mathcal{SN}_{\text{diag}}$. The residue is an invariant of each orbit of $\mathcal{SN}_{\text{fib}}$ under the action of $\text{Diff}_{\text{fib}}(\mathbb{C}^3, 0, \text{Id})$ by conjugacy (see [Bit16]).

The main result of [Bit16] can now be stated as follows:

Theorem 1.5. [Bit16] *Let $Y \in \mathcal{SN}_{\text{diag}}$ be non-degenerate. Then there exists a unique formal fibered diffeomorphism $\hat{\Phi}$ tangent to the identity such that:*

$$\begin{aligned} \hat{\Phi}_*(Y) &= x^2 \frac{\partial}{\partial x} + (-\lambda + a_1 x + c_1(y_1 y_2)) y_1 \frac{\partial}{\partial y_1} \\ (1.6) \quad &+ (\lambda + a_2 x + c_2(y_1 y_2)) y_2 \frac{\partial}{\partial y_2}, \end{aligned}$$

where $\lambda \in \mathbb{C}^*$, $c_1, c_2 \in \mathbb{C}[[v]]$ are formal power series in $v = y_1 y_2$ without constant term and $a_1, a_2 \in \mathbb{C}$ are such that $a_1 + a_2 = \text{res}(Y) \in \mathbb{C} \setminus \mathbb{Q}_{\leq 0}$.

Definition 1.6. The vector field obtained in (1.6) is called the **formal normal form** of Y . The formal fibered diffeomorphism $\hat{\Phi}$ is called the **formal normalizing map** of Y .

The above result is valid for formal objects, without considering problems of convergence. The first main result in this work states that this formal normalizing map is analytic in sectorial domains, under some additional assumptions that we are now going to precise.

Definition 1.7.

- We say that a germ of a vector field X in $(\mathbb{C}^2, 0)$ is **orbitally linear** if

$$X = U(\mathbf{y}) \left(\lambda_1 y_1 \frac{\partial}{\partial y_1} + \lambda_2 y_2 \frac{\partial}{\partial y_2} \right),$$

for some $U(\mathbf{y}) \in \mathbb{C}\{\mathbf{y}\}^\times$ and $(\lambda_1, \lambda_2) \in \mathbb{C}^2$.

- We say that a germ of vector field X in $(\mathbb{C}^2, 0)$ is analytically (*resp.* *formally*) **orbitally linearizable** if X is analytically (*resp.* *formally*) conjugate to an orbitally linear vector field.
- We say that a diagonal doubly-resonant saddle-node $Y \in \mathcal{SN}_{\text{diag}}$ is **div-integrable** if $Y|_{\{x=0\}} \in \chi(\mathbb{C}^2, 0)$ is (analytically) orbitally linearizable.

Remark 1.8. Alternatively we could say that the foliation associated to $Y|_{\{x=0\}}$ is linearizable. Since $Y|_{\{x=0\}}$ is analytic at the origin of \mathbb{C}^2 and has two opposite eigenvalues, it follows from a classical result of Brjuno (see [Mar81]), that $Y|_{\{x=0\}}$ is analytically orbitally linearizable if and only if it is formally orbitally linearizable.

Definition 1.9. We denote by $\mathcal{SN}_{\text{diag}, 0}$ the set of strictly non-degenerate diagonal doubly-resonant saddle-nodes which are div-integrable.

The vector field corresponding to the irregular singularity at infinity in the Painlevé equations $(P_j)_{j=I, \dots, V}$ is orbitally equivalent to an element of $\mathcal{SN}_{\text{fib}, 0}$, for generic values of the parameters (see [Yos85]).

We can now state the first main result of our paper (we refer to subsection 2.3 for the precise definition of *weak 1-summability*).

Theorem 1.10. *Let $Y \in \mathcal{SN}_{\text{diag},0}$ and let $\hat{\Phi}$ (given by Theorem 1.5) be the unique formal fibered diffeomorphism tangent to the identity such that*

$$\begin{aligned}\hat{\Phi}_*(Y) &= x^2 \frac{\partial}{\partial x} + (-\lambda + a_1 x + c_1(y_1 y_2)) y_1 \frac{\partial}{\partial y_1} + (\lambda + a_2 x + c_2(y_1 y_2)) y_2 \frac{\partial}{\partial y_2} \\ &=: Y_{\text{norm}},\end{aligned}$$

where $\lambda \neq 0$ and $c_1(v), c_2(v) \in v\mathbb{C}[[v]]$ are formal power series without constant term. Then:

- (1) *the normal form Y_{norm} is analytic (i.e. $c_1, c_2 \in \mathbb{C}\{v\}$), and it also is div-integrable, i.e. $c_1 + c_2 = 0$;*
- (2) *the formal normalizing map $\hat{\Phi}$ is weakly 1-summable in every direction, except $\arg(\pm\lambda)$;*
- (3) *there exist analytic sectorial fibered diffeomorphisms Φ_+ and Φ_- , (asymptotically) tangent to the identity, defined in sectorial domains of the form $S_+ \times (\mathbb{C}^2, 0)$ and $S_- \times (\mathbb{C}^2, 0)$ respectively, where*

$$\begin{aligned}S_+ &:= \left\{ x \in \mathbb{C} \mid 0 < |x| < r \text{ and } \left| \arg\left(\frac{x}{i\lambda}\right) \right| < \frac{\pi}{2} + \epsilon \right\} \\ S_- &:= \left\{ x \in \mathbb{C} \mid 0 < |x| < r \text{ and } \left| \arg\left(\frac{-x}{i\lambda}\right) \right| < \frac{\pi}{2} + \epsilon \right\}\end{aligned}$$

(for any $\epsilon \in]0, \frac{\pi}{2}[$ and some $r > 0$ small enough), which admit $\hat{\Phi}$ as weak Gevrey-1 asymptotic expansion in these respective domains, and which conjugate Y to Y_{norm} . Moreover Φ_+ and Φ_- are the unique such germs of analytic functions in sectorial domains (see Definition 2.2).

Remark 1.11. Although item 3 above is a straightforward consequence of the *weak 1-summability* (see subsection 2.3) of $\hat{\Phi}$ in item 2 above, we will in fact start by proving item 3 in Corollary 4.2, and show item point 2 in Proposition 5.4.

Definition 1.12. We call Φ_+ and Φ_- the **sectorial normalizing maps** of $Y \in \mathcal{SN}_{\text{diag},0}$.

They are the weak 1-sums of $\hat{\Phi}$ along the respective directions $\arg(i\lambda)$ and $\arg(-i\lambda)$. Notice that Φ_+ and Φ_- are *germs of analytic sectorial fibered diffeomorphisms*, i.e. they are of the form

$$\begin{aligned}\Phi_+ : S_+ \times (\mathbb{C}^2, 0) &\longrightarrow S_+ \times (\mathbb{C}^2, 0) \\ (x, \mathbf{y}) &\longmapsto (x, \Phi_{+,1}(x, \mathbf{y}), \Phi_{+,2}(x, \mathbf{y}))\end{aligned}$$

and

$$\begin{aligned}\Phi_- : S_- \times (\mathbb{C}^2, 0) &\longrightarrow S_- \times (\mathbb{C}^2, 0) \\ (x, \mathbf{y}) &\longmapsto (x, \Phi_{-,1}(x, \mathbf{y}), \Phi_{-,2}(x, \mathbf{y}))\end{aligned}$$

(see section 2. for a precise definition of *germ of analytic sectorial fibered diffeomorphism*). The fact that they are also *(asymptotically) tangent to the identity* means that we have:

$$\Phi_{\pm}(x, \mathbf{y}) = \text{Id}(x, \mathbf{y}) + \mathcal{O}\left(\|(x, \mathbf{y})\|^2\right).$$

In fact, we can prove the uniqueness of the sectorial normalizing maps under weaker assumptions.

Proposition 1.13. *Let φ_+ and φ_- be two germs of sectorial fibered diffeomorphisms in $S_+ \times (\mathbb{C}^2, 0)$ and $S_- \times (\mathbb{C}^2, 0)$ respectively, where S_+ and S_- are as in Theorem 1.10, which are (asymptotically) tangent to the identity and such that*

$$(\varphi_{\pm})_*(Y) = Y_{\text{norm}}.$$

Then, they necessarily coincide with the weak 1-sums Φ_+ and Φ_- defined above.

It is important to say that we will in fact begin with proving the existence of germs of sectorial fibered diffeomorphisms Φ_+ and Φ_- in $S_+ \times (\mathbb{C}^2, 0)$ and $S_- \times (\mathbb{C}^2, 0)$ respectively, which are tangent to the identity and conjugate $Y \in \mathcal{SN}_{\text{fib},0}$ to its normal form (see Corollary 4.2). The proposition above guarantees the uniqueness of such sectorial transforms. It is proved in a second step that Φ_+

and Φ_- admits the formal normalizing map $\hat{\Phi}$ as weak Gevrey-1 asymptotic expansion, which is thus weakly 1-summable.

Remark 1.14. In this paper we prove a theorem of existence of sectorial normalizing map analogous to the classical one due to Hukuhara-Kimura-Matuda for saddle-nodes in $(\mathbb{C}^2, 0)$ [HKM61], generalized later by Stolovitch in any dimension in [Sto96]. Unlike the method based on a fixed point theorem used by these authors, we use a more geometric approach (following the works of Teyssier [Tey03, Tey04]) based on the resolution of an homological equation, by integrating a well chosen 1-form along asymptotic paths. This latter approach turned out to be more efficient to deal with the fact that $Y_{\{x=0\}}$ is not necessarily linearizable. Indeed, if we look at [Sto96] in details, one of the first problem is that in the irregular systems that needs to be solved by a fixed point method (for instance equation (2.7) in the cited paper), the non-linear terms would not be divisible by the “time” variable t in our situation. This would complicate the different estimates that are done later in the cited work. This was the first main new phenomena we have met.

In a forthcoming paper we prove that the sectorial normalizing maps Φ_+, Φ_- in Theorem 1.10 admit in fact the unique formal normalizing map $\hat{\Phi}$ given by Theorem 1.5 as “true” Gevrey-1 asymptotic expansion in $S_+ \times (\mathbb{C}^2, 0)$ and $S_- \times (\mathbb{C}^2, 0)$ respectively. This is done by studying $\Phi_+ \circ (\Phi_-)^{-1}$ in $(S_+ \cap S_-) \times (\mathbb{C}^2, 0)$ (and more generally any germ of sectorial fibered isotropy of Y_{norm} in “narrow” sectorial neighborhoods $(S_+ \cap S_-) \times (\mathbb{C}^2, 0)$ which admits the identity as weak Gevrey-1 asymptotic expansion). In the cited paper we also:

- prove that the formal normalizing map $\hat{\Phi}$ in Theorem 1.10 is in fact 1-summable (and not only *weakly* 1-summable).
- provide a theorem of analytic classification, based on the study over “small” sectors $S_+ \cap S_-$ of the transition maps $\Phi_+ \circ \Phi_-^{-1}$ (also called *Stokes diffeomorphisms*): they are sectorial isotropies of the normal form Y_{norm} which are exponentially close to the identity.

The main difficulty is to establish that such a sectorial isotropy of Y_{norm} over the “narrow” sectors described above is necessarily exponentially close to the identity. This will be done *via* a detailed analysis of these maps in the space of leaves. In fact, this is the second main new difficulty we have met, which is due to the presence of the “resonant” term

$$\frac{c_m (y_1 y_2)^m \log(x)}{x}$$

in the exponential expression of the first integrals of the vector field in normal form. In [Sto96], similar computations are done in subsection 3.4.1. In this part of the paper, infinitely many irregular differential equations appear when identifying terms of same homogeneous degree. The fact that $Y_{\{x=0\}}$ is linear implies that these differential equations are all linear and independent of each others (*i.e.* they are not mixed together). In our situation this is not the case, which yields more complicated computations.

1.3. Painlevé equations: the transversally Hamiltonian case.

In [Yos85] Yoshida shows that a vector field in the class $\mathcal{SN}_{\text{fb},0}$ naturally appears after a suitable compactification (given by the so-called Boutroux coordinates [Bou13]) of the phase space of Painlevé equations $(P_j)_{j=I,\dots,V}$, for generic values of the parameters. In these cases the vector field presents an additional transverse Hamiltonian structure. Let us illustrate these computations in the case of the first Painlevé equation:

$$(P_I) \quad \frac{d^2 z_1}{dt^2} = 6z_1^2 + t \quad .$$

As is well known since Okamoto [Oka80], (P_I) can be seen as a non-autonomous Hamiltonian system

$$\begin{cases} \frac{\partial z_1}{\partial t} = -\frac{\partial H}{\partial z_2} \\ \frac{\partial z_2}{\partial t} = \frac{\partial H}{\partial z_1} \end{cases}$$

with Hamiltonian

$$H(t, z_1, z_2) := 2z_1^3 + tz_1 - \frac{z_2^2}{2}.$$

More precisely, if we consider the standard symplectic form $\omega := dz_1 \wedge dz_2$ and the vector field

$$Z := \frac{\partial}{\partial t} - \frac{\partial H}{\partial z_2} \frac{\partial}{\partial z_1} + \frac{\partial H}{\partial z_1} \frac{\partial}{\partial z_2}$$

induced by (P_I) , then the Lie derivative

$$\mathcal{L}_Z(\omega) = \left(\frac{\partial^2 H}{\partial t \partial z_1} dz_1 + \frac{\partial^2 H}{\partial t \partial z_2} dz_2 \right) \wedge dt = dz_1 \wedge dt$$

belongs to the ideal $\langle dt \rangle$ generated by dt in the exterior algebra $\Omega^*(\mathbb{C}^3)$ of differential forms in variables (t, z_1, z_2) . Equivalently, for any $t_1, t_2 \in \mathbb{C}$ the flow of Z at time $(t_2 - t_1)$ acts as a *symplectomorphism* between fibers $\{t = t_1\}$ and $\{t = t_2\}$.

The weighted compactification given by the Boutroux coordinates [Bou13] defines a chart near $\{t = \infty\}$ as follows:

$$\begin{cases} z_2 = y_2 x^{-\frac{3}{5}} \\ z_1 = y_1 x^{-\frac{2}{5}} \\ t = x^{-\frac{4}{5}} \end{cases}.$$

In the coordinates (x, y_1, y_2) , the vector field Z is transformed, up to a translation $y_1 \leftarrow y_1 + \zeta$ with $\zeta = \frac{i}{\sqrt{6}}$, to the vector field

$$(1.7) \quad \tilde{Z} = -\frac{5}{4x^{\frac{1}{5}}} Y$$

where

$$Y = x^2 \frac{\partial}{\partial x} + \left(-\frac{4}{5} y_2 + \frac{2}{5} x y_1 + \frac{2\zeta}{5} x \right) \frac{\partial}{\partial y_1} + \left(-\frac{24}{5} y_1^2 - \frac{48\zeta}{5} y_1 + \frac{3}{5} x y_2 \right) \frac{\partial}{\partial y_2}.$$

We observe that Y is a strictly non-degenerate doubly-resonant saddle-node as in Definitions 1.1 and 1.3 with residue $\text{res}(Y) = 1$. Furthermore we have:

$$\begin{cases} dt = -\frac{4}{5} 5^{\frac{4}{5}} x^{-\frac{9}{5}} dx \\ dz_1 \wedge dz_2 = \frac{1}{x} (dy_1 \wedge dy_2) + \frac{1}{5x^2} (2y_1 dy_2 - 3y_2 dy_1) \wedge dx \\ \in \frac{1}{x} (dy_1 \wedge dy_2) + \langle dx \rangle \end{cases},$$

where $\langle dx \rangle$ denotes the ideal generated by dx in the algebra of holomorphic forms in $\mathbb{C}^* \times \mathbb{C}^2$. We finally obtain

$$\begin{cases} \mathcal{L}_Y \left(\frac{dy_1 \wedge dy_2}{x} \right) = \frac{1}{5x} (3y_2 dy_1 - (2\zeta + 2y_1) dy_2) \wedge dx \\ \mathcal{L}_Y(dx) = 2x dx \end{cases}.$$

Therefore, both $\mathcal{L}_Y(\omega)$ and $\mathcal{L}_Y(dx)$ are differential forms who lie in the ideal $\langle dx \rangle$, in the algebra of germs of meromorphic 1-forms in $(\mathbb{C}^3, 0)$ with poles only in $\{x = 0\}$. This motivates the following:

Definition 1.15. Consider the rational 1-form

$$\omega := \frac{dy_1 \wedge dy_2}{x}.$$

We say that vector field Y is **transversally Hamiltonian** (with respect to ω and dx) if

$$\mathcal{L}_Y(dx) \in \langle dx \rangle \quad \text{and} \quad \mathcal{L}_Y(\omega) \in \langle dx \rangle.$$

For any open sector $S \subset \mathbb{C}^*$, we say that a germ of sectorial fibered diffeomorphism Φ in $S \times (\mathbb{C}^2, 0)$ is **transversally symplectic** (with respect to ω and dx) if

$$\Phi^*(\omega) \in \omega + \langle dx \rangle$$

(Here $\Phi^*(\omega)$ denotes the pull-back of ω by Φ).

We denote by $\text{Diff}_\omega(\mathbb{C}^3, 0; \text{Id})$ the group of transversally symplectic diffeomorphisms which are tangent to the identity.

Remark 1.16.

- (1) The flow of a transversally Hamiltonian vector field X defines a map between fibers $\{x = x_1\}$ and $\{x = x_2\}$ which sends $\omega|_{x=x_1}$ onto $\omega|_{x=x_2}$, since

$$(\exp(X))^*(\omega) \in \omega + \langle dx \rangle.$$

- (2) A fibered sectorial diffeomorphism Φ is transversally symplectic if and only if $\det(D\Phi) = 1$.

Definition 1.17. A **transversally Hamiltonian doubly-resonant saddle-node** is a transversally Hamiltonian vector field which is conjugate, *via* a transversally symplectic diffeomorphism, to one of the form

$$Y = x^2 \frac{\partial}{\partial x} + \left(-\lambda y_1 + F_1(x, \mathbf{y}) \right) \frac{\partial}{\partial y_1} + \left(\lambda y_2 + F_2(x, \mathbf{y}) \right) \frac{\partial}{\partial y_2},$$

with $\lambda \in \mathbb{C}^*$ and f_1, f_2 analytic in $(\mathbb{C}^3, 0)$ and of order at least 2.

Notice that a transversally Hamiltonian doubly-resonant saddle-node is necessarily strictly non-degenerate (since its residue is always equal to 1), and also div-integrable (see section 3). It follows from Yoshida's work [Yos85] that the doubly-resonant saddle-nodes at infinity in Painlevé equations $(P_j)_{j=I, \dots, V}$ (for generic values of the parameters) all are transversally Hamiltonian.

We recall the second main result from [Bit16].

Theorem 1.18. [Bit16]

Let $Y \in \mathcal{SN}_{\text{diag}}$ be a diagonal doubly-resonant saddle-node which is supposed to be transversally Hamiltonian. Then, there exists a unique formal fibered transversally symplectic diffeomorphism $\hat{\Phi}$, tangent to identity, such that:

$$\begin{aligned} \hat{\Phi}_*(Y) &= x^2 \frac{\partial}{\partial x} + (-\lambda + a_1 x - c(y_1 y_2)) y_1 \frac{\partial}{\partial y_1} + (\lambda + a_2 x + c(y_1 y_2)) y_2 \frac{\partial}{\partial y_2} \\ (1.8) \quad &=: Y_{\text{norm}}, \end{aligned}$$

where $\lambda \in \mathbb{C}^*$, $c(v) \in v\mathbb{C}[[v]]$ a formal power series in $v = y_1 y_2$ without constant term and $a_1, a_2 \in \mathbb{C}$ are such that $a_1 + a_2 = 1$.

As a consequence of Theorem 1.18, Theorem 1.10 we have the following:

Theorem 1.19. *Let Y be a transversally Hamiltonian doubly-resonant saddle-node and let $\hat{\Phi}$ be the unique formal normalizing map given by Theorem 1.18. Then the associate sectorial normalizing maps Φ_+ and Φ_- are also transversally symplectic.*

Proof. Since $\hat{\Phi}$ is weakly 1-summable in $S_\pm \times (\mathbb{C}^2, 0)$, the formal power series $\det(D\hat{\Phi})$ is also weakly 1-summable in $S_\pm \times (\mathbb{C}^2, 0)$, and its asymptotic expansion has to be the constant 1. By uniqueness of the weak 1-sum, we thus have $\det(D\hat{\Phi}_\pm) = 1$. \square

1.4. Outline of the paper.

In section 2, we introduce the different tools we need concerning asymptotic expansion, Gevrey-1 series and 1-summability. We will in particular introduce a notion of “**weak**” 1-summability.

In section 3, we prove Proposition 3.1, which states that we can always formally conjugate a non-degenerate doubly-resonant saddle-node which is also div-integrable to its normal form up to remaining terms of order $O(x^N)$, for all $N \in \mathbb{N}_{>0}$, and the conjugacy is actually 1-summable.

In section 4, we prove that for all $Y \in \mathcal{SN}_{\text{fib},0}$, there exists a pair of sectorial normalizing maps (Φ_+, Φ_-) tangent to the identity which conjugates Y to its normal form Y_{norm} over sectors with opening greater than π and arbitrarily close to 2π (see Corollary 4.2).

In section 5, the uniqueness of the sectorial normalizing maps, stated in Proposition 1.13, is proved thanks to Proposition 5.2. Moreover, we will see that Φ_+ and Φ_- both admit the unique formal normalizing map $\hat{\Phi}$ given by Theorem 1.5 as weak Gevrey-1 asymptotic expansion (see Proposition 5.4).

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2. BACKGROUND

We refer the reader to [MR82, Mal95, RS93, BDM08] for a detailed introduction to the theory of asymptotic expansion, Gevrey series and summability (see also [Sto96] for a useful discussion of these concepts), where one can find the proofs of the classical results we recall (but we do not prove here). We call $x \in \mathbb{C}$ the *independent* variable and $\mathbf{y} := (y_1, \dots, y_n) \in \mathbb{C}^n$, $n \in \mathbb{N}$, the *dependent* variables. As usual we define $\mathbf{y}^{\mathbf{k}} := y_1^{k_1} \dots y_n^{k_n}$ for $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{N}^n$, and $|\mathbf{k}| = k_1 + \dots + k_n$. The notions of asymptotic expansion, Gevrey series and 1-summability presented here are always considered with respect to the independent variable x living in (open) sectors, the dependent variable \mathbf{y} belonging to poly-discs

$$\mathbf{D}(\mathbf{0}, \mathbf{r}) := \{\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{C}^n \mid |y_1| < r_1, \dots, |y_n| < r_n\} \quad ,$$

of poly-radius $\mathbf{r} = (r_1, \dots, r_n) \in (\mathbb{R}_{>0})^n$. Given an open subset

$$\mathcal{U} \subset \mathbb{C}^{n+1} = \{(x, \mathbf{y}) \in \mathbb{C} \times \mathbb{C}^n\}$$

we denote by $\mathcal{O}(\mathcal{U})$ the algebra of holomorphic function in \mathcal{U} . The algebra of germs of analytic functions of m variables $\mathbf{x} := (x_1, \dots, x_m)$ at the origin is denoted by $\mathbb{C}\{\mathbf{x}\}$.

The results recalled in this section are valid when $n = 0$. Some statements for which we do not give a proof can be proved exactly as in the classical case $n = 0$, uniformly in the dependent multi-variable \mathbf{y} . For convenience and homogeneity reasons we will present some classical results not in their original (and more general) form, but rather in more specific cases which we will need. Finally, we will introduce a notion of *weak* Gevrey-1 summability, which we will compare to the classical notion of 1-summability.

2.1. Sectorial germs.

Given $r > 0$, and $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, we denote by $S(r, \alpha, \beta)$ the following open sector:

$$S(r, \alpha, \beta) := \{x \in \mathbb{C} \mid 0 < |x| < r \text{ and } \alpha < \arg(x) < \beta\}.$$

Let $\theta \in \mathbb{R}$, $\eta \in \mathbb{R}_{\geq 0}$ and $n \in \mathbb{N}$.

Definition 2.1. (1) An *x-sectorial neighborhood* (or simply *sectorial neighborhood*) of the origin (in \mathbb{C}^{n+1}) in the direction θ with opening η is an open set $\mathcal{S} \subset \mathbb{C}^{n+1}$ such that

$$\mathcal{S} \supset S\left(r, \theta - \frac{\eta}{2} - \epsilon, \theta + \frac{\eta}{2} + \epsilon\right) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$$

for some $r > 0$, $\mathbf{r} \in (\mathbb{R}_{>0})^n$ and $\epsilon > 0$. We denote by $(\mathcal{S}_{\theta, \eta}, \leq)$ the directed set formed by all such neighborhoods, equipped with the order relation

$$S_1 \leq S_2 \iff S_1 \supset S_2.$$

(2) The algebra of *germs of holomorphic functions in a sectorial neighborhood of the origin in the direction θ with opening η* is the direct limit

$$\mathcal{O}(\mathcal{S}_{\theta, \eta}) := \varinjlim \mathcal{O}(\mathcal{S})$$

with respect to the directed system defined by $\{\mathcal{O}(\mathcal{S}) : \mathcal{S} \in \mathcal{S}_{\theta, \eta}\}$.

We now give the definition of a (*germ of a*) *sectorial diffeomorphism*.

Definition 2.2. (1) Given an element $\mathcal{S} \in \mathcal{S}_{\theta, \eta}$, we denote by $\text{Diff}_{\text{fib}}(\mathcal{S}, \text{Id})$ the set of holomorphic fibered diffeomorphisms of the form

$$\begin{aligned} \Phi : \mathcal{S} &\rightarrow \Phi(\mathcal{S}) \\ (x, \mathbf{y}) &\mapsto (x, \phi_1(x, \mathbf{y}), \phi_2(x, \mathbf{y})) \end{aligned}$$

such that $\Phi(x, \mathbf{y}) - \text{Id}(x, \mathbf{y}) = \mathcal{O}(\|x, \mathbf{y}\|^2)$, as $(x, \mathbf{y}) \rightarrow (0, \mathbf{0})$ in \mathcal{S} .¹

(2) The set of *germs of (fibered) sectorial diffeomorphisms in the direction θ with opening η , tangent to the identity*, is the direct limit

$$\text{Diff}_{\text{fib}}(\mathcal{S}_{\theta, \eta}; \text{Id}) := \varinjlim \text{Diff}_{\text{fib}}(\mathcal{S}, \text{Id})$$

with respect to the directed system defined by $\{\text{Diff}_{\text{fib}}(\mathcal{S}, \text{Id}) : \mathcal{S} \in \mathcal{S}_{\theta, \eta}\}$. We equip $\text{Diff}_{\text{fib}}(\mathcal{S}_{\theta, \eta}; \text{Id})$ with a group structure as follows: given two germs $\Phi, \Psi \in \text{Diff}_{\text{fib}}(\mathcal{S}_{\theta, \eta}; \text{Id})$ we chose corresponding representatives $\Phi_0 \in \text{Diff}_{\text{fib}}(\mathcal{S}, \text{Id})$ and $\Psi_0 \in \text{Diff}_{\text{fib}}(\mathcal{T}, \text{Id})$ with $\mathcal{S}, \mathcal{T} \in \mathcal{S}_{\theta, \eta}$ such that $\mathcal{T} \subset \Phi_0(\mathcal{S})$ and let $\Psi \circ \Phi$ be the germ defined by $\Psi_0 \circ \Phi_0$.²

We will also need the notion of *asymptotic sectors*.

¹This condition implies in particular that $\Phi(\mathcal{S}) \in \mathcal{S}_{\theta, \eta}$.

²One can prove that this definition is independent of the choice of the representatives

Definition 2.3. An (open) asymptotic sector of the origin in the direction θ and with opening η is an open set $S \subset \mathbb{C}$ such that

$$S \in \bigcap_{0 \leq \eta' < \eta} \mathcal{S}_{\theta, \eta'}.$$

We denote by $\mathcal{AS}_{\theta, \eta}$ the set of all such (open) asymptotic sectors.

2.2. Gevrey-1 power series and 1-summability.

2.2.1. Gevrey-1 asymptotic expansions.

In this subsection we fix a formal power series which we write under two forms:

$$\hat{f}(x, \mathbf{y}) = \sum_{k \geq 0} f_k(\mathbf{y}) x^k = \sum_{(j_0, \mathbf{j}) \in \mathbb{N}^{n+1}} f_{j_0, \mathbf{j}} x^{j_0} \mathbf{y}^{\mathbf{j}} \in \mathbb{C}[[x, \mathbf{y}]],$$

using the canonical identification $\mathbb{C}[[x, \mathbf{y}]] = \mathbb{C}[[x]][\mathbf{y}] = \mathbb{C}[\mathbf{y}][[x]]$. We also fix a norm $\|\cdot\|$ in \mathbb{C}^{n+1} .

Definition 2.4.

- A function f analytic in a domain $S(r, \alpha, \beta) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$ admits \hat{f} as asymptotic expansion in the sense of Gérard-Sibuya in this domain if for all closed sub-sector $S' \subset S(r, \alpha, \beta)$ and compact $\mathbf{K} \subset \mathbf{D}(\mathbf{0}, \mathbf{r})$, for all $N \in \mathbb{N}$, there exists a constant $C_{S', K, N} > 0$ such that:

$$\left| f(x, \mathbf{y}) - \sum_{j_0 + j_1 + \dots + j_n \leq N} f_{j_0, \mathbf{j}} x^{j_0} \mathbf{y}^{\mathbf{j}} \right| \leq C_{S', K, N} \|(x, \mathbf{y})\|^{N+1}$$

for all $(x, \mathbf{y}) \in S' \times K$.

- A function f analytic in a domain $S(r, \alpha, \beta) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$ admits \hat{f} as asymptotic expansion (with respect to x) if for all $k \in \mathbb{N}$, $f_k(\mathbf{y})$ is analytic in $\mathbf{D}(\mathbf{0}, \mathbf{r})$, and if for all closed sub-sector $S' \subset S(r, \alpha, \beta)$, compact subset $\mathbf{K} \subset \mathbf{D}(\mathbf{0}, \mathbf{r})$ and $N \in \mathbb{N}$, there exists $A_{S', K, N} > 0$ such that:

$$\left| f(x, \mathbf{y}) - \sum_{k \geq 0} f_k(\mathbf{y}) x^k \right| \leq A_{S', K, N} |x|^{N+1}$$

for all $(x, \mathbf{y}) \in S' \times K$.

- An analytic function f in a sectorial domain $S(r, \alpha, \beta) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$ admits \hat{f} as Gevrey-1 asymptotic expansion in this domain, if for all $k \in \mathbb{N}$, $f_k(\mathbf{y})$ is analytic in $\mathbf{D}(\mathbf{0}, \mathbf{r})$, and if for all closed sub-sector $S' \subset S(r, \alpha, \beta)$, there exists $A, C > 0$ such that:

$$\left| f(x, \mathbf{y}) - \sum_{k=0}^{N-1} f_k(\mathbf{y}) x^k \right| \leq AC^N (N!) |x|^N$$

for all $N \in \mathbb{N}$ and $(x, \mathbf{y}) \in S' \times \mathbf{D}(\mathbf{0}, \mathbf{r})$.

Remark 2.5.

- (1) If a function admits \hat{f} as Gevrey-1 asymptotic expansion in $S(r, \alpha, \beta) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$, then it also admits \hat{f} as asymptotic expansion.
- (2) If a function admits \hat{f} as asymptotic expansion in $S(r, \alpha, \beta) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$, then it also admits \hat{f} as asymptotic expansion in the sense of Gérard-Sibuya.
- (3) An asymptotic expansion (in any of the different senses described above) is unique.

As a consequence of Stirling formula, we have the following characterization for functions admitting 0 as Gevrey-1 asymptotic expansion.

Proposition 2.6. The set of analytic functions admitting 0 as Gevrey-1 asymptotic expansion at the origin in a sectorial domain $S(r, \alpha, \beta) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$ is exactly the set of analytic functions f in

$S(r, \alpha, \beta) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$ such that for all closed sub-sector $S' \subset S(r, \alpha, \beta)$ and all compact $\mathbf{K} \subset \mathbf{D}(\mathbf{0}, \mathbf{r})$, there exist $A_{S', K}, B_{S', K} > 0$ such that:

$$|f(x, \mathbf{y})| \leq A_{S', K} \exp\left(-\frac{B_{S', K}}{|x|}\right).$$

We say that such a function is exponentially flat at the origin in the corresponding domain.

2.2.2. Borel transform and Gevrey-1 power series.

Definition 2.7.

- We define the Borel transform $\mathcal{B}(\hat{f})$ of \hat{f} as:

$$\mathcal{B}(\hat{f})(t, \mathbf{y}) := \sum_{k \geq 0} \frac{f_k(\mathbf{y})}{k!} t^k.$$

- We say that \hat{f} is Gevrey-1 if $\mathcal{B}(\hat{f})$ is convergent in a neighborhood of the origin in $\mathbb{C} \times \mathbb{C}^n$. Notice that in this case the $f_k(\mathbf{y})$, $k \geq 0$, are all analytic in a same polydisc $\mathbf{D}(\mathbf{0}, \mathbf{r})$, of poly-radius $\mathbf{r} = (r_1, \dots, r_n) \in (\mathbb{R}_{>0})^n$, so that $\mathcal{B}(\hat{f})$ is analytic in $\mathbf{D}(0, \rho) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$, for some $\rho > 0$. Possibly by reducing $\rho, r_1, \dots, r_n > 0$, we can assume that $\mathcal{B}(\hat{f})$ is bounded in $\mathbf{D}(0, \rho) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$.

Remark 2.8.

- (1) If a sectorial function f admits \hat{f} for Gevrey-1 asymptotic expansion as in Definition 2.4 then \hat{f} is a Gevrey-1 formal power series.
- (2) The set of all Gevrey-1 formal power series is an algebra closed under partial derivatives $\frac{\partial}{\partial x}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}$.

Remark 2.9. For technical reasons we will sometimes need to use another definition of the Borel transform, that is:

$$\tilde{\mathcal{B}}(\hat{f})(t, \mathbf{y}) := \sum_{k \geq 0} f_{k+1}(\mathbf{y}) \frac{t^k}{k!}.$$

The first definition we gave has the advantage of being “directly” invertible (*via* the Laplace transform) for all 1-summable formal power series (see next subsection), but behaves not so good with respect to the product. On the contrary, the second definition will be “directly” invertible only for 1-summable formal power series with null constant term (otherwise a translation is needed). However, the advantage of the second Borel transform is that it changes a product into a convolution product:

$$\tilde{\mathcal{B}}(\hat{f}\hat{g}) = \left(\tilde{\mathcal{B}}(\hat{f}) * \tilde{\mathcal{B}}(\hat{g})\right),$$

where the convolution product of two analytic functions $h_1 h_2$ is defined by

$$(h_1 * h_2)(t, \mathbf{y}) := \int_0^t h_1(s) h_2(s - t) ds.$$

The property of being Gevrey-1 or not does not depend on the choice of the definition we take for the Borel transform.

2.2.3. Directional 1-summability and Borel-Laplace summation.

Definition 2.10. Given $\theta \in \mathbb{R}$ and $\delta > 0$, we define the infinite sector in the direction θ with opening δ as the set

$$\mathcal{A}_{\theta,\delta}^\infty := \left\{ t \in \mathbb{C}^* \mid |\arg(t) - \theta| < \frac{\delta}{2} \right\}.$$

We say that \hat{f} is 1-summable in the direction $\theta \in \mathbb{R}$, if the following three conditions holds:

- \hat{f} is a Gevrey-1 formal power series;
- $\mathcal{B}(\hat{f})$ can be analytically continued to a domain of the form $\mathcal{A}_{\theta,\delta}^\infty \times \mathbf{D}(\mathbf{0}, \mathbf{r})$;
- there exists $\lambda > 0, M > 0$ such that:

$$\forall (t, \mathbf{y}) \in \mathcal{A}_{\theta,\delta}^\infty \times \mathbf{D}(\mathbf{0}, \mathbf{r}), \quad \left| \mathcal{B}(\hat{f})(t, \mathbf{y}) \right| \leq M \exp(\lambda |t|).$$

In this case we set $\Delta_{\theta,\delta,\rho} := \mathcal{A}_{\theta,\delta}^\infty \cup \mathbf{D}(0, \rho)$ and

$$\left\| \hat{f} \right\|_{\lambda,\theta,\delta,\rho,\mathbf{r}} := \sup_{(t,\mathbf{y}) \in \Delta_{\theta,\delta,\rho} \times \mathbf{D}(\mathbf{0}, \mathbf{r})} \left| \mathcal{B}(\hat{f})(t, \mathbf{y}) \exp(-\lambda |t|) \right|.$$

If the domain is clear from the context we will simply write $\left\| \hat{f} \right\|_\lambda$.

Remark 2.11.

- (1) For fixed $(\lambda, \theta, \delta, \rho, \mathbf{r})$ as above, the set $\mathfrak{B}_{\lambda,\theta,\delta,\rho,\mathbf{r}}$ of formal power series \hat{f} 1-summable in the direction θ and such that $\left\| \hat{f} \right\|_{\lambda,\theta,\delta,\rho,\mathbf{r}} < +\infty$ is a Banach vector space for the norm $\|\cdot\|_{\lambda,\theta,\delta,\rho,\mathbf{r}}$. We simply write $(\mathfrak{B}_\lambda, \|\cdot\|_\lambda)$ when there is no ambiguity.
- (2) We will also need a norm well-adapted to the second Borel transform \tilde{B} (cf. Remark 2.9), that is:

$$\left\| \hat{f} \right\|_{\lambda,\theta,\delta,\rho,\mathbf{r}}^{\text{bis}} := \sup_{(t,\mathbf{y}) \in \Delta_{\theta,\delta,\rho} \times \mathbf{D}(\mathbf{0}, \mathbf{r})} \left| \mathcal{B}(\hat{f})(t, \mathbf{y}) (1 + \lambda^2 |t|^2) \exp(-\lambda |t|) \right|.$$

We write then $\mathfrak{B}_{\lambda,\theta,\delta,\rho,\mathbf{r}}^{\text{bis}}$ the set space of formal power series \hat{f} which are 1-summable in the direction θ and such that $\left\| \hat{f} \right\|_{\lambda,\theta,\delta,\rho,\mathbf{r}}^{\text{bis}} < +\infty$.

- (3) If $\lambda' \geq \lambda$, then $\mathfrak{B}_{\lambda,\theta,\delta,\rho,\mathbf{r}} \subset \mathfrak{B}_{\lambda',\theta,\delta,\rho,\mathbf{r}}$ and $\mathfrak{B}_{\lambda,\theta,\delta,\rho,\mathbf{r}}^{\text{bis}} \subset \mathfrak{B}_{\lambda',\theta,\delta,\rho,\mathbf{r}}^{\text{bis}}$.

Proposition 2.12 ([BDM08, Proposition 4.]). *If $\hat{f}, \hat{g} \in \mathfrak{B}_{\lambda,\theta,\delta,\rho,\mathbf{r}}^{\text{bis}}$, then $\hat{f}\hat{g} \in \mathfrak{B}_{\lambda,\theta,\delta,\rho,\mathbf{r}}^{\text{bis}}$ and:*

$$\left\| \hat{f}\hat{g} \right\|_{\lambda,\theta,\delta,\rho,\mathbf{r}}^{\text{bis}} \leq \frac{4\pi}{\lambda} \left\| \hat{f} \right\|_{\lambda,\theta,\delta,\rho,\mathbf{r}}^{\text{bis}} \left\| \hat{g} \right\|_{\lambda,\theta,\delta,\rho,\mathbf{r}}^{\text{bis}}.$$

Remark 2.13. If $\lambda \geq 4\pi$, then $\|\cdot\|_{\lambda,\theta,\delta,\rho,\mathbf{r}}^{\text{bis}}$ is a sub-multiplicative norm, i.e.

$$\left\| \hat{f}\hat{g} \right\|_{\lambda,\theta,\delta,\rho,\mathbf{r}}^{\text{bis}} \leq \left\| \hat{f} \right\|_{\lambda,\theta,\delta,\rho,\mathbf{r}}^{\text{bis}} \left\| \hat{g} \right\|_{\lambda,\theta,\delta,\rho,\mathbf{r}}^{\text{bis}}.$$

Definition 2.14. Let g be analytic in a domain and $\mathcal{A}_{\theta,\delta}^\infty \times \mathbf{D}(\mathbf{0}, \mathbf{r})$ and let $\lambda > 0, M > 0$ such that

$$\forall (t, \mathbf{y}) \in \mathcal{A}_{\theta,\delta}^\infty \times \mathbf{D}(\mathbf{0}, \mathbf{r}), \quad |g(t, \mathbf{y})| \leq M \exp(\lambda |t|).$$

We define the *Laplace transform* of g in the direction θ as:

$$\mathcal{L}_\theta(g)(x, \mathbf{y}) := \int_{e^{i\theta}\mathbb{R}_{>0}} g(t, \mathbf{y}) \exp\left(-\frac{t}{x}\right) \frac{dt}{x},$$

which is absolutely convergent for all $x \in \mathbb{C}$ with $\Re\left(\frac{e^{i\theta}}{x}\right) > \lambda$ and $\mathbf{y} \in \mathbf{D}(\mathbf{0}, \mathbf{r})$, and analytic with respect to (x, \mathbf{y}) in this domain.

Remark 2.15. As for the Borel transform, there also exists another definition of the Laplace transform, that is:

$$\tilde{\mathcal{L}}_\theta(g)(x, \mathbf{y}) := \int_{e^{i\theta}\mathbb{R}_{>0}} g(t, \mathbf{y}) \exp\left(-\frac{t}{x}\right) dt .$$

Proposition 2.16. *A formal power series $\hat{f} \in \mathbb{C}[[x, \mathbf{y}]]$ is 1-summable in the direction θ if and only if there exists a germ of a sectorial holomorphic function $f_\theta \in \mathcal{O}(\mathcal{S}_{\theta, \pi})$ which admits \hat{f} as Gevrey-1 asymptotic expansion in some $\mathcal{S} \in \mathcal{S}_{\theta, \pi}$. Moreover, f_θ is unique (as a germ in $\mathcal{O}(\mathcal{S}_{\theta, \pi})$) and in particular*

$$f_\theta = \mathcal{L}_\theta\left(\mathcal{B}\left(\hat{f}\right)\right) .$$

The function (germ) f_θ is called the 1-sum of \hat{f} in the direction θ .

Remark 2.17. With the second definitions of Borel and Laplace transforms given above, we have a similar result for formal power series of the form $\hat{f}(x, \mathbf{y}) = \sum_k f_k(\mathbf{y}) x^k$ with:

$$f_\theta = \tilde{\mathcal{L}}_\theta\left(\tilde{\mathcal{B}}\right)\left(\hat{f}\right) + \hat{f}(0, \mathbf{y}) .$$

We recall the following well-known result.

Lemma 2.18. *The set $\Sigma_\theta \subset \mathbb{C}[[x, \mathbf{y}]]$ of 1-summable power series in the direction θ is an algebra closed under partial derivatives. Moreover the map*

$$\begin{aligned} \Sigma_\theta &\longrightarrow \mathcal{O}(\mathcal{S}_{\theta, \pi}) \\ \hat{f} &\longmapsto f_\theta \end{aligned}$$

is an injective morphism of differential algebras.

Definition 2.19. A formal power series $\hat{f} \in \mathbb{C}[[x, \mathbf{y}]]$ is called 1-summable if it is 1-summable in all but a finite number of directions, called *Stokes directions*. In this case, if $\theta_1, \dots, \theta_k \in \mathbb{R}/2\pi\mathbb{Z}$ are the possible Stokes directions, we say that \hat{f} is 1-summable except for $\theta_1, \dots, \theta_k$.

More generally, we say that an m -uple $(f_1, \dots, f_m) \in \mathbb{C}[[x, \mathbf{y}]]^m$ is Gevrey-1 (resp. 1-summable in direction θ) if this property holds for each component $f_j, j = 1, \dots, m$. Similarly, a formal vector field (or diffeomorphism) is said to be Gevrey-1 (resp. 1-summable in direction θ) if each one of its components has this property.

The following classical result deals with composition of 1-summable power series (an elegant way to prove it is to use an important theorem of Ramis-Sibuya).

Proposition 2.20. *Let $\hat{\Phi}(x, \mathbf{y}) \in \mathbb{C}[[x, \mathbf{y}]]$ be 1-summable in directions θ and $\theta - \pi$, and let $\Phi_+(x, \mathbf{y})$ and $\Phi_-(x, \mathbf{y})$ be its 1-sums directions θ and $\theta - \pi$ respectively. Let also $\hat{f}_1(x, \mathbf{z}), \dots, \hat{f}_n(x, \mathbf{z})$ be 1-summable in directions $\theta, \theta - \pi$, and $f_{1,+}, \dots, f_{n,+}$, and $f_{1,-}, \dots, f_{n,-}$ be their 1-sums in directions θ and $\theta - \pi$ respectively. Assume that*

$$(2.1) \quad \hat{f}_j(0, \mathbf{0}) = 0, \text{ for all } j = 1, \dots, n .$$

Then

$$\hat{\Psi}(x, \mathbf{z}) := \hat{\Phi}\left(x, \hat{f}_1(x, \mathbf{z}), \dots, \hat{f}_n(x, \mathbf{z})\right)$$

is 1-summable in directions $\theta, \theta - \pi$, and its 1-sum in the corresponding direction is

$$\Psi_\pm(x, \mathbf{z}) := \Phi_\pm(x, f_{1,\pm}(x, \mathbf{z}), \dots, f_{n,\pm}(x, \mathbf{z})) ,$$

which is a germ of a sectorial holomorphic function in this direction.

Consider \hat{Y} a formal singular vector field at the origin and a formal fibered diffeomorphism $\hat{\varphi} : (x, \mathbf{y}) \mapsto (x, \hat{\varphi}(x, \mathbf{y}))$. Assume that both \hat{Y} and $\hat{\varphi}$ are 1-summable in directions θ and $\theta - \pi$, for some $\theta \in \mathbb{R}$, and denote by Y_+, Y_- (resp. φ_+, φ_-) their 1-sums in directions θ and $\theta - \pi$ respectively. As a consequence of Proposition 2.20 and Lemma 2.18, we can state the following:

Corollary 2.21. *Under the assumptions above, $\hat{\varphi}_* (\hat{Y})$ is 1-summable in both directions θ and $\theta - \pi$, and its 1-sums in these directions are $\varphi_+ (Y_+)$ and $\varphi_- (Y_-)$ respectively.*

2.3. Weak Gevrey-1 power series and weak 1-summability.

We present here a weaker notion of 1-summability that we will also need. Any function $f(x, \mathbf{y})$ analytic in a domain $\mathcal{U} \times \mathbf{D}(\mathbf{0}, \mathbf{r})$, where $\mathcal{U} \subset \mathbb{C}$ is open, and bounded in any domain $\mathcal{U} \times \overline{\mathbf{D}}(\mathbf{0}, \mathbf{r}')$ with $r'_1 < r_1, \dots, r'_n < r_n$, can be written

$$(2.2) \quad f(x, \mathbf{y}) = \sum_{\mathbf{j} \in \mathbb{N}^n} F_{\mathbf{j}}(x) \mathbf{y}^{\mathbf{j}} \quad ,$$

where for all $\mathbf{j} \in \mathbb{N}^n$, $F_{\mathbf{j}}$ is analytic and bounded on \mathcal{U} , and defined *via* the Cauchy formula:

$$F_{\mathbf{j}}(x) = \frac{1}{(2i\pi)^n} \int_{|z_1|=r'_1} \cdots \int_{|z_n|=r'_n} \frac{f(x, \mathbf{z})}{(z_1)^{j_1+1} \cdots (z_n)^{j_n+1}} dz_n \cdots dz_1 \quad .$$

Notice that the convergence of the function series above is uniform in every compact with respect to x and \mathbf{y} .

In the same way, any formal power series $\hat{f}(x, \mathbf{y}) \in \mathbb{C}[[x, \mathbf{y}]]$ can be written as

$$\hat{f}(x, \mathbf{y}) = \sum_{\mathbf{j} \in \mathbb{N}^n} \hat{F}_{\mathbf{j}}(x) \mathbf{y}^{\mathbf{j}} \quad .$$

Definition 2.22.

- The formal power series \hat{f} is said to be **weakly Gevrey-1** if for all $\mathbf{j} \in \mathbb{N}^n$, $\hat{F}_{\mathbf{j}}(x) \in \mathbb{C}[[x]]$ is a Gevrey-1 formal power series.
- A function

$$f(x, \mathbf{y}) = \sum_{\mathbf{j} \in \mathbb{N}^n} F_{\mathbf{j}}(x) \mathbf{y}^{\mathbf{j}}$$

analytic and bounded in a domain $S(r, \alpha, \beta) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$, admits \hat{f} as **weak Gevrey-1 asymptotic expansion** in $x \in S(r, \alpha, \beta)$, if for all $\mathbf{j} \in \mathbb{N}^n$, $F_{\mathbf{j}}$ admits $\hat{F}_{\mathbf{j}}$ as Gevrey-1 asymptotic expansion in $S(r, \alpha, \beta)$.

- The formal power series \hat{f} is said to be **weakly 1-summable in the direction $\theta \in \mathbb{R}$** , if the following conditions hold:
 - for all $\mathbf{j} \in \mathbb{N}^n$, $\hat{F}_{\mathbf{j}}(x) \in \mathbb{C}[[x]]$ is 1-summable in the direction θ , whose 1-sum in the direction θ is denoted by $F_{\mathbf{j}, \theta}$;
 - the series $f_{\theta}(x, \mathbf{y}) := \sum_{\mathbf{j} \in \mathbb{N}^n} F_{\mathbf{j}, \theta}(x) \mathbf{y}^{\mathbf{j}}$ defines a germ of a sectorial holomorphic function in a sectorial neighborhood attached to the origin in the direction θ with opening greater than π .

In this case, $f_{\theta}(x, \mathbf{y})$ is called **the weak 1-sum of \hat{f} in the direction θ** .

As a consequence to the classical theory of summability and Gevrey asymptotic expansions, we immediately have the following:

- Lemma 2.23.** (1) *The weak Gevrey-1 asymptotic expansion of an analytic function in a domain $S(r, \alpha, \beta) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$ is unique.*
- (2) *The weak 1-sum of a weak 1-summable formal power series in the direction θ , is unique as a germ in $\mathcal{O}(\mathcal{S}_{\theta, \pi})$.*
- (3) *The set $\Sigma_{\theta}^{(\text{weak})} \subset \mathbb{C}[[x, \mathbf{y}]]$ of weakly 1-summable power series in the direction θ is an algebra closed under partial derivatives. Moreover the map*

$$\begin{aligned} \Sigma_{\theta}^{(\text{weak})} &\longrightarrow \mathcal{O}(\mathcal{S}_{\theta, \pi}) \\ \hat{f} &\longmapsto f_{\theta} \end{aligned}$$

is an injective morphism of differential algebras.

The following proposition is an analogue of Proposition 2.20 for weak 1-summable formal power series, with the a stronger condition instead of (2.1).

Proposition 2.24. *Let*

$$\hat{\Phi}(x, \mathbf{y}) = \sum_{\mathbf{j} \in \mathbb{N}^n} \hat{\Phi}_{\mathbf{j}}(x) \mathbf{y}^{\mathbf{j}} \in \mathbb{C}[[x, \mathbf{y}]]$$

and

$$\hat{f}^{(k)}(x, \mathbf{z}) = \sum_{\mathbf{j} \in \mathbb{N}^n} \hat{F}_{\mathbf{j}}^{(k)}(x) \mathbf{z}^{\mathbf{j}} \in \mathbb{C}[[x, \mathbf{z}]] ,$$

for $k = 1, \dots, n$, be $n+1$ formal power series which are weakly 1-summable in directions θ and $\theta - \pi$. Let us denote by $\Phi_+, f_+^{(1)}, \dots, f_+^{(n)}$ (resp. $\Phi_-, f_-^{(1)}, \dots, f_-^{(n)}$) their respective weak 1-sums in the direction θ (resp. $\theta - \pi$). Assume that $\hat{F}_{\mathbf{0}}^{(k)} = 0$ for all $k = 1, \dots, n$. Then,

$$\hat{\Psi}(x, \mathbf{z}) := \hat{\Phi}\left(x, \hat{f}^{(1)}(x, \mathbf{z}), \dots, \hat{f}^{(n)}(x, \mathbf{z})\right)$$

is weakly 1-summable directions θ and $\theta - \pi$, and its 1-sum in the corresponding direction is

$$\Psi_{\pm}(x, \mathbf{z}) = \Phi_{\pm}\left(x, f_{\pm}^{(1)}(x, \mathbf{z}), \dots, f_{\pm}^{(n)}(x, \mathbf{z})\right) ,$$

which is a germ of a sectorial holomorphic function in this direction with opening π .

Proof. First of all,

$$\hat{\Psi}(x, \mathbf{z}) := \hat{\Phi}\left(x, \hat{f}^{(1)}(x, \mathbf{z}), \dots, \hat{f}^{(n)}(x, \mathbf{z})\right)$$

is well defined as formal power series since for all $k = 1, \dots, n$, $\hat{F}_{\mathbf{0}}^{(k)} = 0$. It is also clear that

$$\Psi_{\pm}(x, \mathbf{z}) := \Phi_{\pm}\left(x, f_{\pm}^{(1)}(x, \mathbf{z}), \dots, f_{\pm}^{(n)}(x, \mathbf{z})\right)$$

is an analytic in a domain $\mathcal{S}_+ \in \mathcal{S}_{\theta, \pi}$ (resp. $\mathcal{S}_- \in \mathcal{S}_{\theta - \pi, \pi}$), because $f_{\pm}^{(k)}(x, \mathbf{0}) = 0$ for all $k = 1, \dots, n$. Finally, we check that Ψ_{\pm} admits $\hat{\Psi}$ as weak Gevrey-1 asymptotic expansion in \mathcal{S}_{\pm} . Indeed:

$$\begin{aligned} \Psi_{\pm}(x, \mathbf{z}) &= \Phi_{\pm}\left(x, f_{\pm}^{(1)}(x, \mathbf{z}), \dots, f_{\pm}^{(n)}(x, \mathbf{z})\right) \\ &= \sum_{\mathbf{j} \in \mathbb{N}^n} (\Phi_{\mathbf{j}})_{\pm}(x) \left(f_{\pm}^{(1)}(x, \mathbf{z})\right)^{j_1} \dots \left(f_{\pm}^{(n)}(x, \mathbf{z})\right)^{j_n} \\ &= \sum_{\mathbf{j} \in \mathbb{N}^n} (\Phi_{\mathbf{j}})_{\pm}(x) \left(\sum_{|\mathbf{l}| \geq 1} \left(F_{\mathbf{l}}^{(1)}\right)_{\pm}(x) \mathbf{z}^{\mathbf{l}}\right)^{j_1} \dots \\ &\quad \dots \left(\sum_{|\mathbf{l}| \geq 1} \left(F_{\mathbf{l}}^{(n)}\right)_{\pm}(x) \mathbf{z}^{\mathbf{l}}\right)^{j_n} \\ &= \sum_{\mathbf{j} \in \mathbb{N}^n} (\Psi_{\mathbf{j}})_{\pm}(x) \mathbf{y}^{\mathbf{j}} \end{aligned}$$

where for all $\mathbf{j} \in \mathbb{N}^n$, $(\Psi_{\mathbf{j}})_{\pm}(x)$ is obtained as a finite number of additions and products of the $(\Phi_{\mathbf{k}})_{\pm}, \left(F_{\mathbf{k}}^{(1)}\right)_{\pm}, \dots, \left(F_{\mathbf{k}}^{(n)}\right)_{\pm}$, $|\mathbf{k}| \leq |\mathbf{j}|$. The same computation is valid for the associated formal power series, and allows us to define the $\hat{\Psi}_{\mathbf{j}}(x)$, for all $\mathbf{j} \in \mathbb{N}^n$. Then, each $(\Psi_{\mathbf{j}})_{\pm}$ has $\hat{\Psi}_{\mathbf{j}}$ as Gevrey-1 asymptotic expansion in \mathcal{S}_{\pm} . \square

As a consequence of Proposition 2.24 and Lemma 2.23, we have an analogue version of Corollary (2.21) in the weak 1-summable case. Consider \hat{Y} a formal singular vector field at the origin and a formal fibered diffeomorphism $\hat{\varphi} : (x, \mathbf{y}) \mapsto \left(x, \hat{\phi}(x, \mathbf{y})\right)$ such that $\hat{\phi}(x, \mathbf{0}) = \mathbf{0}$. Assume that both \hat{Y}

and $\hat{\varphi}$ are weakly 1-summable in directions θ and $\theta - \pi$, for some $\theta \in \mathbb{R}$, and denote by Y_+, Y_- (resp. φ_+, φ_-) their weak 1-sums in directions θ and $\theta - \pi$ respectively.

Corollary 2.25. *Under the assumptions above, $\hat{\varphi}_* (\hat{Y})$ is weakly 1-summable in both directions θ and $\theta - \pi$, and its 1-sums in these directions are $\varphi_+ (Y_+)$ and $\varphi_- (Y_-)$ respectively.*

2.4. Weak 1-summability versus 1-summability.

As in the previous subsection, let a formal power series $\hat{f}(x, \mathbf{y}) \in \mathbb{C}[[x, \mathbf{y}]]$ which is written as

$$\hat{f}(x, \mathbf{y}) = \sum_{\mathbf{j} \in \mathbb{N}^n} \hat{F}_{\mathbf{j}}(x) \mathbf{y}^{\mathbf{j}},$$

so that its Borel transform is

$$\mathcal{B}(\hat{f})(t, \mathbf{y}) = \sum_{\mathbf{j} \in \mathbb{N}^n} \mathcal{B}(\hat{F}_{\mathbf{j}})(t) \mathbf{y}^{\mathbf{j}}.$$

The next lemma is immediate.

Lemma 2.26.

- (1) *The power series $\mathcal{B}(\hat{f})(t, \mathbf{y})$ is convergent in a neighborhood of the origin in \mathbb{C}^{n+1} if and only if the $\mathcal{B}(\hat{F}_{\mathbf{j}}), \mathbf{j} \in \mathbb{N}^n$, are all analytic and bounded in a same disc $D(0, \rho)$, $\rho > 0$, and if there exists $B, L > 0$ such that for all $\mathbf{j} \in \mathbb{N}^n$, $\sup_{t \in D(0, \rho)} |\mathcal{B}(\hat{F}_{\mathbf{j}})(t)| \leq L.B^{|\mathbf{j}|}$.*
- (2) *If 1. is satisfied, then $\mathcal{B}(\hat{f})$ can be analytically continued to a domain $\mathcal{A}_{\theta, \delta}^\infty \times \mathbf{D}(\mathbf{0}, \mathbf{r})$ if and only if for all $\mathbf{j} \in \mathbb{N}^n$, $\mathcal{B}(\hat{F}_{\mathbf{j}})$ can be analytically continued to $\mathcal{A}_{\theta, \delta}^\infty$ and if for all compact $K \subset \mathcal{A}_{\theta, \delta}^\infty$, there exists $B, L > 0$ such that for all $\mathbf{j} \in \mathbb{N}^n$, $\sup_{t \in K} |\mathcal{B}(\hat{F}_{\mathbf{j}})(t)| \leq L.B^{|\mathbf{j}|}$.*
- (3) *If and 1. and 2. are satisfied, then there exists $\lambda, M > 0$ such that:*

$$\forall (t, \mathbf{y}) \in \mathcal{A}_{\theta, \delta}^\infty \times \mathbf{D}(\mathbf{0}, \mathbf{r}), \quad |\mathcal{B}(\hat{f})(t, \mathbf{y})| \leq M \cdot \exp(\lambda |t|)$$

if and only if there exists $\lambda, L, B > 0$ such that for all $\mathbf{j} \in \mathbb{N}^n$,

$$\forall t \in \mathcal{A}_{\theta, \delta}^\infty, \quad |\mathcal{B}(\hat{F}_{\mathbf{j}})(t)| \leq L.B^{|\mathbf{j}|} \exp(\lambda |t|).$$

Remark 2.27.

- (1) Condition 1. above states that the formal power series \hat{f} is Gevrey-1.
- (2) As usual, there exists an equivalent lemma for the second definitions of the Borel transform (see Remark 2.9).

The following corollary gives a link between 1-summability and weak 1-summability.

Corollary 2.28. *Let*

$$\hat{f}(x, \mathbf{y}) = \sum_{\mathbf{j} \in \mathbb{N}^n} \hat{F}_{\mathbf{j}}(x) \mathbf{y}^{\mathbf{j}} \in \mathbb{C}[[x, \mathbf{y}]]$$

be a formal power series. Then, \hat{f} is 1-summable in the direction $\theta \in \mathbb{R}$, of 1-sum $f \in \mathcal{O}(\mathcal{S}_{\theta, \pi})$, if and only if the following two conditions hold:

- *\hat{f} is weakly 1-summable in the direction θ ;*
- *there exists λ, δ, ρ such that for all $\mathbf{j} \in \mathbb{N}^n$, $\|\hat{F}_{\mathbf{j}}\|_{\lambda, \theta, \delta, \rho} < \infty$ and the power series $\sum_{\mathbf{j} \in \mathbb{N}^n} \|\hat{F}_{\mathbf{j}}\|_{\lambda, \theta, \delta, \rho} \mathbf{y}^{\mathbf{j}}$ is convergent near the origin of \mathbb{C}^n .*

Proof. This is an immediate consequence of Lemma 2.26. □

Remark 2.29. We can replace the norm $\|\cdot\|_{\lambda, \theta, \delta, \rho}$ by $\|\cdot\|_{\lambda, \theta, \delta, \rho}^{\text{bis}}$ in the second point of the above corollary.

Notice that there exists formal power series which are weakly 1-summable in some direction but which are not Gevrey-1: for instance, the series

$$\hat{f} := \sum_j \hat{F}_j(x) y^j ,$$

where for all $j \in \mathbb{N}$, $\hat{F}_j(x)$ is such that $\mathcal{B}\left(\hat{F}_j\right)(t) = \frac{1}{t + \frac{1}{j}}$, is weakly 1-summable in the direction $0 \in \mathbb{R}$, but is *not* Gevrey-1, since $\mathcal{B}\left(\hat{F}_j\right)$ has a pole in every $-\frac{1}{j} \xrightarrow{j \rightarrow +\infty} 0$.

2.5. Some useful tools on 1-summability of solutions of singular linear differential equations.

For future reuse, we give here two results on the 1-summability of formal solutions to some singular linear differential equations with 1-summable right hand side, which generalize (and precise) a similar result proved in [MR82] (Proposition p. 126). The authors use a norm $\|\cdot\|_\beta$, but we will need to use a norm $\|\cdot\|_\beta^{\text{bis}}$ later (in the proof of Proposition 3.14).

Proposition 2.30. *Let \hat{b} be a formal power series 1-summable in the direction θ ; consider a domain $\Delta_{\theta,\delta,\rho}$ as in Definition 2.10. Let us denote by b_θ its 1-sum in this direction θ . Let us also fix $\alpha, k \in \mathbb{C}$.*

- (1) *Assume $\|\hat{b}\|_\beta^{\text{bis}} < +\infty$ and that $k \in \mathbb{C} \setminus \{0\}$ is such that $d_k := \text{dist}(-k, \Delta_{\theta,\delta,\rho}) > 0$ and*

$$\beta d_k > C |\alpha k| ,$$

where $C > 0$ is a constant large enough, independent from parameters $k, \beta, \theta, \delta, \rho$ (for instance, one can take $C = \frac{2\exp(2)}{5} + 5$). Then, the irregular singular differential equation

$$(2.3) \quad x^2 \frac{da}{dx}(x) + (1 + \alpha x) ka(x) = \hat{b}(x)$$

has a unique formal solution \hat{a} such that $\hat{a}(0) = \frac{1}{k} \hat{b}(0)$. Moreover, \hat{a} is 1-summable in the direction θ , and

$$(2.4) \quad \|\hat{a}\|_\beta \leq \frac{\beta}{\beta d_k - C |\alpha k|} \|\hat{b}\|_\beta .$$

Finally, the 1-sum a_θ of \hat{a} in the direction θ is the only solution to

$$x^2 \frac{da_\theta}{dx}(x) + (1 + \alpha x) ka_\theta(x) = b_\theta(x)$$

which is bounded in some $S_{\theta,\pi} \in \mathcal{S}_{\theta,\pi}$.

- (2) *Assume $\|\hat{b}\|_\beta < +\infty$ and that $\Re(k) > 0$. Then the regular singular differential equation*

$$(2.5) \quad x \frac{da}{dx}(x) + ka(x) = \hat{b}(x)$$

admits a unique formal solution \hat{a} which is also 1-summable in the direction θ , of 1-sum a_θ . Moreover, a_θ is the only germ of solution to the differential equation

$$x \frac{da}{dx}(x) + ka(x) = b_\theta(x)$$

which is bounded in some $S_{\theta,\pi} \in \mathcal{S}_{\theta,\pi}$.

Proof.

- (1) Since \hat{b} is 1-summable in the direction θ , we can choose $\rho > 0$ and $\delta > 0$ such that $\tilde{\mathcal{B}}(\hat{b})$ can be analytically continued to (and is bounded in) any domain of the form $\Delta_{\theta,\delta,\rho} \cap \overline{\mathcal{D}}(0, R)$, $R > 0$.

Let us apply the Borel transform $\tilde{\mathcal{B}}$ to equation (2.3): we obtain

$$(2.6) \quad (t+k) \tilde{\mathcal{B}}(\hat{a})(t) + \alpha k \int_0^t \tilde{\mathcal{B}}(\hat{a})(s) ds = \tilde{\mathcal{B}}(\hat{b})(t) \quad .$$

The derivative with respect to t of this equation shows that $\tilde{\mathcal{B}}(\hat{a})$ is solution of a linear differential equation, with only one (regular) singularity at $t = -k$ (but this singularity is not in $\Delta_{\theta,\delta,\rho}$ by assumption):

$$(t+k) \frac{d\tilde{\mathcal{B}}(\hat{a})}{dt}(t) + (1+\alpha k) \tilde{\mathcal{B}}(\hat{a})(t) = \frac{d\tilde{\mathcal{B}}(\hat{b})}{dt}(t) \quad .$$

Since $\tilde{\mathcal{B}}(\hat{b})$ can be analytically continued to $\Delta_{\theta,\delta,\rho}$, the same goes for $\frac{d\tilde{\mathcal{B}}(\hat{b})}{dt}(t)$ and then for $\tilde{\mathcal{B}}(\hat{a})$.

Since $\tilde{\mathcal{B}}(\hat{a})(0) = \frac{\tilde{\mathcal{B}}(\hat{b})(0)}{k} = \frac{\hat{b}'(0)}{k}$, we can write:

$$\begin{aligned} \tilde{\mathcal{B}}(\hat{a})(t) &= (t+k)^{-1-\alpha k} \left(\hat{b}'(0) \cdot k^{\alpha k} + \int_0^t \frac{d\tilde{\mathcal{B}}(\hat{b})}{ds}(s) \cdot (s+k)^{\alpha k} ds \right) \\ &= (t+k)^{-1-\alpha k} \left(\hat{b}'(0) \cdot k^{\alpha k} + \tilde{\mathcal{B}}(\hat{b})(t) \cdot (t+k)^{\alpha k} - \tilde{\mathcal{B}}(\hat{b})(0) \cdot k^{\alpha k} \right. \\ &\quad \left. - \alpha k \int_0^t \tilde{\mathcal{B}}(\hat{b})(s) \cdot (s+k)^{\alpha k-1} ds \right) \\ &= (t+k)^{-1-\alpha k} \left(\tilde{\mathcal{B}}(\hat{b})(t) \cdot (t+k)^{\alpha k} \right. \\ &\quad \left. - \alpha k \int_0^t \tilde{\mathcal{B}}(\hat{b})(s) \cdot (s+k)^{\alpha k-1} ds \right) \\ \tilde{\mathcal{B}}(\hat{a}) &= \frac{\tilde{\mathcal{B}}(\hat{b})(t)}{(t+k)} - \alpha k \cdot (t+k)^{-1-\alpha k} \int_0^t \tilde{\mathcal{B}}(\hat{b})(s) \cdot (s+k)^{\alpha k-1} ds . \end{aligned}$$

The fact that $\tilde{\mathcal{B}}(\hat{b})$ is bounded in any domain of the form $\Delta_{\theta,\delta,\rho} \cap \overline{\mathcal{D}}(0, R)$, $R > 0$, implies that the same goes for $\tilde{\mathcal{B}}(\hat{a})$. Let us prove inequality (2.4). For all $R > 0$, for all Gevrey-1 series $\hat{f} \in \mathbb{C}[[x, \mathbf{y}]]$ such that $\mathcal{B}(\hat{f})$ can be analytically continued to $\Delta_{\theta,\delta,r}$, we set:

$$\left\| \hat{f} \right\|_{\beta, R}^{\text{bis}} := \sup_{t \in \Delta_{\theta,\delta,\rho} \cap \overline{\mathcal{D}}(0, R)} \left\{ \left| \tilde{\mathcal{B}}(\hat{f})(t) \left(1 + \beta^2 |t|^2 \right) \exp(-\beta |t|) \right| \right\} \in \mathbb{R} \cup \{\infty\} .$$

Notice that $\left\| \hat{f} \right\|_{\beta}^{\text{bis}} = \sup_{R>0} \left\{ \left\| \hat{f} \right\|_{\beta, R}^{\text{bis}} \right\}$ for all \hat{f} as above, and that for all $R > 0$, $\|\hat{a}\|_{\beta, R}^{\text{bis}} < +\infty$, since $\tilde{\mathcal{B}}(\hat{a})$ is bounded in any domain of the form $\Delta_{\theta,\delta,\rho} \cap \overline{\mathcal{D}}(0, R)$. Fix some $R > 0$, and let $t \in \Delta_{\theta,\delta,\rho} \cap \overline{\mathcal{D}}(0, R)$. From equation (2.6) we obtain

$$\tilde{\mathcal{B}}(\hat{a})(t) = \frac{1}{(t+k)} \left(\tilde{\mathcal{B}}(\hat{b})(t) - \alpha k \int_0^t \tilde{\mathcal{B}}(\hat{a})(s) ds \right)$$

an then

$$\begin{aligned} \left| \tilde{\mathcal{B}}(\hat{a})(t) \right| &\leq \frac{1}{|t+k|} \left[\left\| \hat{b} \right\|_{\beta}^{\text{bis}} \frac{\exp(\beta |t|)}{1 + \beta^2 |t|^2} + |\alpha k| \cdot \|\hat{a}\|_{\beta, R}^{\text{bis}} \int_0^{|t|} \frac{\exp(\beta u)}{1 + \beta^2 u^2} du \right] \\ &\leq \frac{1}{d_k} \frac{\exp(\beta |t|)}{1 + \beta^2 |t|^2} \left[\left\| \hat{b} \right\|_{\beta}^{\text{bis}} + |\alpha k| \|\hat{a}\|_{\beta, R}^{\text{bis}} \frac{C}{\beta} \right] , \end{aligned}$$

with $C = \frac{2 \exp(2)}{5} + 5$. Here we use the following:

Fact 2.31. *There exists a constant $C > 0$ (e.g. $C = \frac{2 \exp(2)}{5} + 5$), such that for all $\beta > 0$, we have:*

$$\forall t \geq 0, \int_0^t \frac{\exp(\beta u)}{1 + \beta^2 u^2} du \leq \frac{C \exp(\beta t)}{\beta (1 + \beta^2 t^2)}.$$

Proof. Let $F : u \mapsto \frac{\exp(\beta u)}{1 + \beta^2 u^2}$, for $u \geq 0$. For $t \in [0, \frac{2}{\beta}]$, we have:

$$\int_0^t F(u) du \leq \frac{\exp(2)}{5} \cdot \frac{2}{\beta},$$

since F is an increasing function over \mathbb{R}_+ :

$$F'(u) = \beta F(u) \cdot \frac{(1 - \beta u)^2}{1 + \beta^2 u^2} \geq 0.$$

Moreover for all $t \geq 0$, we have $F(t) \geq F(0) = 1$. Hence for all $t \in [0, \frac{2}{\beta}]$:

$$\int_0^t F(u) du \leq \frac{\exp(2)}{5} \cdot \frac{2}{\beta} \cdot F(t).$$

For $t \geq \frac{2}{\beta}$, the following inequality holds:

$$\int_0^t F(u) du \leq \frac{\exp(2)}{5} \cdot \frac{2}{\beta} F(t) + \int_{\frac{2}{\beta}}^t F(u) du.$$

In addition, if $u \geq \frac{\beta}{2}$, then:

$$\frac{(1 - \beta u)^2}{1 + \beta^2 u^2} \geq \frac{1}{5},$$

Therefore, for all $u \geq \frac{\beta}{2}$:

$$F'(u) = \beta F(u) \cdot \frac{(1 - \beta u)^2}{1 + \beta^2 u^2} \geq \frac{\beta}{5} F(u).$$

Hence:

$$\begin{aligned} \int_0^t F(u) du &\leq \int_0^{\frac{2}{\beta}} F(u) du + \int_{\frac{2}{\beta}}^t F(u) du \\ &\leq \frac{\exp(2)}{5} \cdot \frac{2}{\beta} F(t) + \frac{5}{\beta} \int_{\frac{2}{\beta}}^t F'(u) du \\ &\leq \frac{F(t)}{5\beta} \cdot (2 \exp(2) + 25). \end{aligned}$$

□

Let us go back to the proof of the lemma. Finally, we have:

$$\|\hat{a}\|_{\beta,R}^{\text{bis}} \leq \frac{1}{d_k} \left[\|\hat{b}\|_{\beta}^{\text{bis}} + \frac{C \cdot |\alpha k| \cdot \|\hat{a}\|_{\beta,R}^{\text{bis}}}{\beta} \right],$$

and consequently:

$$\|\hat{a}\|_{\beta,R}^{\text{bis}} \leq \frac{\beta}{\beta d_k - C |\alpha k|} \|\hat{b}\|_{\beta}^{\text{bis}}.$$

As a conclusion:

$$\|\hat{a}\|_{\beta}^{\text{bis}} \leq \frac{\beta}{\beta d_k - C |\alpha k|} \|\hat{b}\|_{\beta}^{\text{bis}},$$

and a_θ is the 1-sum of \hat{a} in the direction θ . (2) Let us write $\hat{b}(x) = \sum_{j \geq 0} b_j x^j$. A direct computation

shows that the only formal solution to equation (2.5) is $\hat{a}(x) = \sum_{j \geq 0} a_j x^j$ where for all $j \in \mathbb{N}$, $a_j = \frac{b_j}{j+k}$: it exists since $k \notin \mathbb{Z}_{\leq 0}$, and then $k+j \neq 0$. In particular, we see immediately that \hat{a} is Gevrey-1, because the same goes for \hat{b} . In other words, the Borel transform $\mathcal{B}(\hat{a})$ is analytic in some disc $D(0, \rho)$, $\rho > 0$. In $D(0, \rho)$, $\mathcal{B}(\hat{a})$ satisfies:

$$t \frac{d\mathcal{B}(\hat{a})}{dt}(t) + k\mathcal{B}(\hat{a})(t) = \mathcal{B}(\hat{b})(t).$$

The general solution near the origin to this equation is

$$y(t) = \frac{c}{t^k} + \frac{1}{t^k} \int_0^t \mathcal{B}(\hat{b})(s) s^{k-1} ds, \quad c \in \mathbb{C}.$$

In particular, the only solution analytic in $D(0, \rho)$ is the one with $c = 0$, i.e.

$$\mathcal{B}(\hat{a})(t) = \frac{1}{t^k} \int_0^t \mathcal{B}(\hat{b})(s) s^{k-1} ds.$$

Since $\mathcal{B}(\hat{b})$ can be analytically continued to an infinite domain that have denoted by $\Delta_{\theta, \delta, \rho}$ bisected by $\mathbb{R}_+ e^{i\theta}$ (because \hat{b} is 1-summable in the direction θ), $\mathcal{B}(\hat{a})$ can also be analytically continued to the same domain. Moreover, there exists $\beta > 0$ such that $\|\hat{b}\|_\beta < +\infty$, i.e. $\forall t \in \Delta_{\theta, \delta, \rho}$:

$$|\mathcal{B}(\hat{b})(t)| \leq \|\hat{b}\|_\beta \exp(\beta |t|).$$

Thus, for all $t \in \Delta_{\theta, \delta, \rho}$, we have:

$$\begin{aligned} |\exp(-\beta |t|) \mathcal{B}(\hat{a})(t)| &\leq \frac{1}{|t|^k} \int_0^{|t|} |\exp(-\beta |t|) \mathcal{B}(\hat{b})(se^{i \arg(t)})| s^{k-1} e^{i(k-1) \arg(t)} ds \\ &\leq \frac{1}{|t|^{\Re(k)}} \int_0^{|t|} |\exp(-\beta s)| \mathcal{B}(\hat{b})(se^{i \arg(t)})| s^{\Re(k)-1} ds \\ &\leq \frac{\|\hat{b}\|_\beta}{|t|^{\Re(k)}} \int_0^{|t|} s^{\Re(k)-1} ds \\ &= \frac{\|\hat{b}\|_\beta}{\Re(k)}. \end{aligned}$$

Thus, \hat{a} is 1-summable in the direction θ . □

3. 1-SUMMABLE PREPARATION UP TO ANY ORDER N

The aim of this section is to prove that we can always formally conjugate a non-degenerate doubly-resonant saddle-node, which is also div-integrable, to its normal form up to a remainder of order $O(x^N)$ for every $N \in \mathbb{N}_{>0}$. Moreover, we prove that this conjugacy is in fact 1-summable in every direction $\theta \neq \arg(\pm \lambda)$, hence analytic over sectorial domains of opening at least π .

Proposition 3.1. *Let $Y \in \mathcal{SN}_{\text{diag}}$ be a non-degenerate diagonal doubly-resonant saddle-node which is div-integrable, with $D_0 Y = \text{diag}(0, -\lambda, \lambda)$, $\lambda \neq 0$. Then, for all $N \in \mathbb{N}_{>0}$, there exists a formal fibered*

diffeomorphism $\Psi^{(N)} \in \widehat{\text{Diff}}_{\text{fib}}(\mathbb{C}^3, 0; \text{Id})$ tangent to the identity and 1-summable in every direction $\theta \neq \arg(\pm\lambda)$ such that:

$$\begin{aligned} \left(\Psi^{(N)}\right)_*(Y) &= x^2 \frac{\partial}{\partial x} + \left(\left(-\left(\lambda + d^{(N)}(y_1 y_2) \right) + a_1 x \right) + x^N F_1^{(N)}(x, \mathbf{y}) \right) y_1 \frac{\partial}{\partial y_1} \\ &\quad + \left(\left(\lambda + d^{(N)}(y_1 y_2) + a_2 x \right) + x^N F_2^{(N)}(x, \mathbf{y}) \right) y_2 \frac{\partial}{\partial y_2} \\ &=: Y^{(N)}, \end{aligned}$$

where $\lambda \in \mathbb{C}^*$, $(a_1 + a_2) = \text{res}(Y) \in \mathbb{C} \setminus \mathbb{Q}_{\leq 0}$, $d^{(N)}(v) \in v\mathbb{C}\{v\}$ is an analytic germ vanishing at the origin, and $F_1^{(N)}, F_2^{(N)} \in \mathbb{C}[[x, \mathbf{y}]]$ are 1-summable in the direction θ , and of order at least one with respect to \mathbf{y} . Moreover, one can choose $d^{(2)} = \dots = d^{(N)}$ for all $N \geq 2$.

Definition 3.2. A vector field $Y^{(N)}$ as is the proposition above is said to be *normalized up to order N* .

Remark 3.3.

- (1) Observe that this result does not require the more restrictive assumption of being “strictly non-degenerate” (i.e. $\Re(a_1 + a_2) > 0$).
- (2) As a consequence of Corollary 2.21, the 1-sum $\Psi_\theta^{(N)}$ of $\Psi^{(N)}$ in the direction θ is a germ of sectorial fibered diffeomorphism tangent to the identity, i.e. $\Psi_\theta^{(N)} \in \text{Diff}_{\text{fib}}(\mathcal{S}_{\theta, \pi}; \text{Id})$, which conjugates Y to the 1-sum $Y_\theta^{(N)}$ of $Y^{(N)}$ in the direction θ .

In order to prove this result we will proceed in several steps and use after each step Proposition 2.20 and Corollary 2.21 in order to prove the 1-summability in every direction $\theta \neq \arg(\pm\lambda)$ of the different objects. First, we will normalize analytically the vector field restricted to $\{x = 0\}$. Then, we will straighten the formal separatrix to $\{y_1 = y_2 = 0\}$ in suitable coordinates. Next, we will simplify the linear terms with respect to \mathbf{y} . After that, we will straighten two invariant hypersurfaces to $\{y_1 = 0\}$ and $\{y_2 = 0\}$. Finally, we will conjugate the vector field to its final normal form up to remaining terms of order $O(x^N)$.

3.1. Analytic normalization on the hyperplane $\{x = 0\}$.

3.1.1. Transversally Hamiltonian versus div-integrable.

We start by proving that an element of $\mathcal{SN}_{\text{diag}}$ which is transversally Hamiltonian is necessarily div-integrable.

Proposition 3.4. *If $Y \in \mathcal{SN}_{\text{diag}}$ is transversally Hamiltonian, then Y is div-integrable.*

Proof. Let us consider more generally a diagonal doubly-resonant saddle-node $Y \in \mathcal{SN}_{\text{diag}}$ such that $Y|_{\{x=0\}}$ is a Hamiltonian vector field with respect to $dy_1 \wedge dy_2$ (this is the case if Y is transversally Hamiltonian): there exists a Hamiltonian $H(\mathbf{y}) = \lambda y_1 y_2 + O(\|\mathbf{y}\|^3) \in \mathbb{C}\{\mathbf{y}\}$, such that

$$Y = x^2 \frac{\partial}{\partial x} + \left(\left(-\frac{\partial H}{\partial y_2} + x F_1(x, \mathbf{y}) \right) \frac{\partial}{\partial y_1} + \left(\frac{\partial H}{\partial y_1} + x F_2(x, \mathbf{y}) \right) \frac{\partial}{\partial y_2} \right),$$

where $F_1, F_2 \in \mathbb{C}\{x, \mathbf{y}\}$ are vanishing at the origin. If we define $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{C})$ and $\nabla H := {}^t(DH)$, then $Y|_{\{x=0\}} = J\nabla H$. According to the Morse lemma for holomorphic functions, there exists a germ of an analytic change of coordinates $\varphi \in \text{Diff}(\mathbb{C}^2, 0)$ given by

$$\mathbf{y} = (y_1, y_2) \mapsto \varphi(\mathbf{y}) = \left(y_1 + O(\|\mathbf{y}\|^2), y_2 + O(\|\mathbf{y}\|^2) \right),$$

such that $\tilde{H}(\mathbf{y}) := H(\varphi^{-1}(\mathbf{y})) = y_1 y_2$. Let us now recall a trivial result from linear algebra.

Fact. *Let $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{C})$, and $P \in M_2(\mathbb{C})$. Then, $PJ^t P = \det(P) J$.*

As a consequence we have:

Corollary 3.5. *Let $H \in \mathbb{C}\{\mathbf{y}\}$ be a germ of an analytic function at 0, $Y_0 := J\nabla H$ the associated Hamiltonian vector field in \mathbb{C}^2 (for the usual symplectic form $dy_1 \wedge dy_2$), and an analytic diffeomorphism near the origin denoted by φ . Then:*

$$\varphi_*(Y_0) := (D\varphi \circ \varphi^{-1}) \cdot (Y_0 \circ \varphi^{-1}) = \det(D\varphi \circ \varphi^{-1}) J\nabla \tilde{H} \quad ,$$

where $\tilde{H} := H \circ \varphi^{-1}$.

As a conclusion we have proved that Y is div-integrable. \square

3.1.2. General case.

Now we prove how to normalize the restriction to $\{x = 0\}$ of a div-integrable element of $\mathcal{SN}_{\text{diag}}$.

Proposition 3.6. *Let $Y \in \mathcal{SN}_{\text{diag}}$ be div-integrable. Then, there exists $\psi \in \text{Diff}_{\text{fib}}(\mathbb{C}^3, 0; \text{Id})$ of the form*

$$\psi : (x, \mathbf{y}) \mapsto \left(x, y_1 + O(\|\mathbf{y}\|^2), y_2 + O(\|\mathbf{y}\|^2) \right)$$

such that

$$\psi_*(Y) = x^2 \frac{\partial}{\partial x} + (- (\lambda + d(v)) y_1 + x T_1(x, \mathbf{y})) \frac{\partial}{\partial y_1} + ((\lambda + d(v)) y_2 + x T_2(x, \mathbf{y})) \frac{\partial}{\partial y_2} \quad ,$$

with $v := y_1 y_2$, $d(v) \in v\mathbb{C}\{v\}$ and $T_1, T_2 \in \mathbb{C}\{x, \mathbf{y}\}$ vanishing at the origin.

Proof. By assumption, and according to a theorem due to Brjuno (cf [Mar81]), up to a first transformation analytic at the origin in \mathbb{C}^2 , we can suppose that

$$Y_{|\{x=0\}} = (\lambda + h(\mathbf{y})) \left(-y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2} \right) \quad .$$

Then, it remains to apply the following lemma to $Y_{|\{x=0\}}$. \square

Lemma 3.7. *Let Y_0 be a germ of analytic vector field in $(\mathbb{C}^2, 0)$ of the form*

$$Y_0 = (\lambda + h(\mathbf{y})) \left(-y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2} \right) \quad ,$$

with $h \in \mathbb{C}\{\mathbf{y}\}$ vanishing at the origin. Then there exists $\phi \in \text{Diff}(\mathbb{C}^2, 0, \text{Id})$ such that

$$\phi_*(Y_0) = (\lambda + d(v)) \left(-y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2} \right) \quad ,$$

with $v := y_1 y_2$ and $d \in v\mathbb{C}\{v\}$.

Remark 3.8. In other words, we have removed every non-resonant term in $h(\mathbf{y})$. In fact, we re-obtain here a particular case (with one vector field in dimension 2) of the principal result in [Sto97] (which is itself inspired of Vey's works).

Proof. We claim that ϕ can be chosen of the form

$$\phi(\mathbf{y}) = \left(y_1 e^{-\gamma(\mathbf{y})}, y_2 e^{\gamma(\mathbf{y})} \right) \quad ,$$

for a conveniently chosen $\gamma \in \mathbb{C}\{\mathbf{y}\}$. Indeed, let us study how such a diffeomorphism acts on Y_0 . Let us write $U := (\lambda + h(v))$ and $L := \left(-y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2} \right)$, such that $Y_0 = UL$. An easy computation shows:

$$\begin{aligned} \phi_*(Y_0) &= \phi_*(UL) \\ &= ([U \cdot (1 - \mathcal{L}_L(\gamma))] \circ \phi^{-1}) L \quad , \end{aligned}$$

where \mathcal{L}_L is the Lie derivative of associate to L . We want to find γ such that the unit

$$D := [U (1 - \mathcal{L}_L(\gamma))] \circ \phi^{-1}$$

is free from *non-resonant* terms, *i.e.* is of the form

$$D = \lambda + d(y_1 y_2) = \lambda + \sum_{k \geq 1} d_k (y_1 y_2)^k .$$

Notice that if a unit $W = \sum_{k \geq 0} W_k (y_1 y_2)^k \mathbb{C}\{\mathbf{y}\}^\times$ is free from non-resonant terms, then:

$$\begin{cases} W \circ \phi^{-1} = W \\ \mathcal{L}_L(W) = 0 \end{cases} .$$

Thus, let us split both U and γ in a “resonant” and a “non-resonant” part:

$$\begin{cases} U = U_{\text{res}} + U_{\text{n-res}} \\ \gamma = \gamma_{\text{res}} + \gamma_{\text{n-res}} \end{cases}$$

where

$$\begin{cases} U_{\text{n-res}} = \sum_{k_1 \neq k_2} U_{k_1, k_2} y_1^{k_1} y_2^{k_2} \\ U_{\text{res}} = \sum_k U_{k, k} (y_1 y_2)^k \\ \gamma_{\text{n-res}} = \sum_{k_1 \neq k_2} \gamma_{k_1, k_2} y_1^{k_1} y_2^{k_2} \\ \gamma_{\text{res}} = \sum_k \gamma_{k, k} (y_1 y_2)^k \end{cases} .$$

Then the non-resonant terms of $U(1 - \mathcal{L}_L(\gamma))$ are

$$(U_{\text{n-res}} - (U_{\text{n-res}} + U_{\text{res}}) \mathcal{L}_L(\gamma_{\text{n-res}})) \circ \phi^{-1} .$$

Hence, the partial differential equation we want to solve is:

$$\mathcal{L}_L(\gamma) = \frac{U_{\text{n-res}}}{U_{\text{res}} + U_{\text{n-res}}} .$$

One sees immediately that this equation admit an analytic solution (and even infinitely many solutions) $\gamma \in \mathbb{C}\{\mathbf{y}\}$, since the unit $U \in \mathbb{C}\{\mathbf{y}\}$ is analytic. \square

3.2. 1-summable simplification of the “dependent” affine part.

We are concerned by studying vector fields of the form

$$(3.1) \quad Y = x^2 \frac{\partial}{\partial x} + (-\lambda y_1 + f_1(x, \mathbf{y})) \frac{\partial}{\partial y_1} + (\lambda y_2 + f_2(x, \mathbf{y})) \frac{\partial}{\partial y_2} ,$$

with

$$\begin{cases} f_1(x, \mathbf{y}) = -d(y_1 y_2) y_1 + x T_1(x, \mathbf{y}) \\ f_2(x, \mathbf{y}) = d(y_1 y_2) y_2 + x T_2(x, \mathbf{y}) \end{cases} ,$$

where $d(v) \in v\mathbb{C}\{v\}$ and $T_1, T_2 \in \mathbb{C}\{x, \mathbf{y}\}$ are of order at least one.

Proposition 3.9. *Let $Y \in \mathcal{SN}_{\text{diag}}$ be a doubly-resonant saddle-node of the form*

$$Y = x^2 \frac{\partial}{\partial x} + (-\lambda y_1 + f_1(x, \mathbf{y})) \frac{\partial}{\partial y_1} + (\lambda y_2 + f_2(x, \mathbf{y})) \frac{\partial}{\partial y_2} ,$$

where $f_1, f_2 \in \mathbb{C}\{x, \mathbf{y}\}$ are such that $f_1(x, \mathbf{y}), f_2(x, \mathbf{y}) = \mathcal{O}(\|(x, \mathbf{y})\|^2)$. Then there exist formal power series $\hat{y}_1, \hat{y}_2, \hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}_1, \hat{\beta}_2 \in x\mathbb{C}[[x]]$ which are 1-summable in every direction $\theta \neq \arg(\pm\lambda)$, such that the formal fibered diffeomorphism

$$\hat{\Phi} : (x, y_1, y_2) \mapsto \left(x, \hat{y}_1(x) + (1 + \hat{\alpha}_1(x)) y_1 + \hat{\beta}_1(x) y_2, \hat{y}_2(x) + \hat{\alpha}_2(x) y_1 + (1 + \hat{\beta}_1(x)) y_2 \right) ,$$

(which is tangent to the identity and 1-summable in every direction $\theta \neq \arg(\pm\lambda)$) conjugates Y to

$$\hat{\Phi}_*(Y) = x^2 \frac{\partial}{\partial x} + \left((-\lambda + a_1 x) y_1 + \hat{F}_1(x, \mathbf{y}) \right) \frac{\partial}{\partial y_1} + \left((\lambda + a_2 x) y_2 + \hat{F}_2(x, \mathbf{y}) \right) \frac{\partial}{\partial y_2} ,$$

where $a_1, a_2 \in \mathbb{C}$ and $\hat{F}_1, \hat{F}_2 \in \mathbb{C}[[x, \mathbf{y}]]$ are of order at least 2 with respect to \mathbf{y} , and 1-summable in every direction $\theta \neq \arg(\pm\lambda)$.

Remark 3.10. Notice that $\hat{\Phi}|_{\{x=0\}} = \text{Id}$, so that $\hat{F}_i(0, \mathbf{y}) = f_i(0, \mathbf{y})$ for $i = 1, 2$. Moreover, the residue of $\hat{\Phi}_*(Y)$ is $a_1 + a_2$.

The proof of Proposition 3.9 is postponed to subsection 3.2.2.

3.2.1. Technical lemmas on irregular differential systems.

Lemma 3.11. *There exists a pair of formal power series $(\hat{y}_1(x), \hat{y}_2(x)) \in (x\mathbb{C}[[x]])^2$ which are 1-summable in every direction $\theta \neq \arg(\pm\lambda)$, such that the formal diffeomorphism given by*

$$\hat{\Phi}_1(x, y_1, y_2) = (x, y_1 - \hat{y}_1(x), y_2 - \hat{y}_2(x)),$$

(which is 1-summable in every direction $\theta \neq \arg(\pm\lambda)$) conjugates Y in (3.1) to

$$(3.2) \quad \hat{Y}_1(x, \mathbf{y}) = x^2 \frac{\partial}{\partial x} + (-\lambda y_1 + \hat{g}_1(x, \mathbf{y})) \frac{\partial}{\partial y_1} + (\lambda y_2 + \hat{g}_2(x, \mathbf{y})) \frac{\partial}{\partial y_2},$$

where \hat{g}_1, \hat{g}_2 are formal power series of order at least 2 such that $\hat{g}_1(x, \mathbf{0}) = \hat{g}_2(x, \mathbf{0}) = 0$, and are 1-summable in every direction $\theta \neq \arg(\pm\lambda)$.

In other words, in the new coordinates, the curve given by $(y_1, y_2) = (0, 0)$ is invariant by the flow of the vector field, and contains the origin in its closure: it is usually called the (formal, 1-summable) center manifold.

Proof. This is an immediate consequence of an important theorem by Ramis and Sibuya on the summability of formal solutions to irregular differential systems [RS89]. This theorem proves the existence and the 1-summability in every direction $\theta \neq \arg(\pm\lambda)$, of \hat{y}_1 and \hat{y}_2 : $(\hat{y}_1(x), \hat{y}_2(x))$ is defined as the unique formal solution to

$$\begin{cases} x^2 \frac{dy_1}{dx} = -\lambda y_1(x) + f_1(x, y_1(x), y_2(x)) \\ x^2 \frac{dy_2}{dx} = \lambda y_2(x) + f_2(x, y_1(x), y_2(x)) \end{cases},$$

such that $(\hat{y}_1(0), \hat{y}_2(0)) = (0, 0)$. The 1-summability of \hat{g}_1 and \hat{g}_2 comes from Proposition 2.20. \square

The next step is aimed at changing to linear terms with respect to \mathbf{y} in “diagonal” form.

Lemma 3.12. *There exists a pair of formal power series $(\hat{p}_1, \hat{p}_2) \in (\mathbb{C}[[x]])^2$ which are 1-summable in every direction $\theta \neq \arg(\pm\lambda)$, such that the formal fibered diffeomorphism given by*

$$\hat{\Phi}_2(x, y_1, y_2) = (x, y_1 + x\hat{p}_2(x)y_2, y_2 + x\hat{p}_1(x)y_1),$$

(which is tangent to the identity and 1-summable in every direction $\theta \neq \arg(\pm\lambda)$) conjugates \hat{Y}_1 in (3.2), to

$$(3.3) \quad \begin{aligned} \hat{Y}_2(x, \mathbf{y}) = & x^2 \frac{\partial}{\partial x} + \left((-\lambda + x\hat{a}_1(x)) y_1 + \hat{H}_1(x, \mathbf{y}) \right) \frac{\partial}{\partial y_1} \\ & + \left((\lambda + x\hat{a}_2(x)) y_2 + \hat{H}_2(x, \mathbf{y}) \right) \frac{\partial}{\partial y_2}, \end{aligned}$$

where $\hat{a}_1, \hat{a}_2, \hat{H}_1, \hat{H}_2$ are formal power series which are 1-summable in every direction $\theta \neq \arg(\pm\lambda)$ and \hat{H}_1, \hat{H}_2 are of order at least 2 with respect to \mathbf{y} .

Proof. Let us write

$$\begin{cases} \hat{g}_1(x, \mathbf{y}) = x\hat{b}_1(x)y_1 + x\hat{c}_1(x)y_2 + \hat{G}_1(x, \mathbf{y}) \\ \hat{g}_2(x, \mathbf{y}) = x\hat{c}_2(x)y_1 + x\hat{b}_2(x)y_2 + \hat{G}_2(x, \mathbf{y}) \end{cases},$$

where $\hat{b}_1, \hat{b}_2, \hat{c}_1, \hat{c}_2, \hat{G}_1, \hat{G}_2$ are formal power series 1-summable in the direction $\theta \neq \arg(\pm\lambda)$, such that \hat{G}_1 and \hat{G}_2 are of order at least 2 with respect to \mathbf{y} . Let us consider the following irregular differential system naturally associated to \hat{Y}_1 :

$$(3.4) \quad x^2 \frac{d\mathbf{z}}{dx}(x) = \hat{\mathbf{B}}(x) \mathbf{z}(x) + \hat{\mathbf{G}}(x, \mathbf{z}(x)) \quad ,$$

where

$$\hat{\mathbf{B}}(x) = \begin{pmatrix} -\lambda + x\hat{b}_1(x) & x\hat{c}_1(x) \\ x\hat{c}_2(x) & \lambda + x\hat{b}_2(x) \end{pmatrix}, \quad \hat{\mathbf{G}}(x, \mathbf{z}(x)) = \begin{pmatrix} \hat{G}_1(x, \mathbf{z}(x)) \\ \hat{G}_2(x, \mathbf{z}(x)) \end{pmatrix}.$$

We are looking for

$$\hat{\mathbf{P}}(x) = \begin{pmatrix} 1 & x\hat{p}_2(x) \\ x\hat{p}_1(x) & 1 \end{pmatrix} \in \text{GL}_2(\mathbb{C}[[x]]),$$

where \hat{p}_1, \hat{p}_2 are 1-summable formal power series in x every direction $\theta \neq \arg(\pm\lambda)$, such that the linear transformation given by $\mathbf{z}(x) = \hat{\mathbf{P}}(x) \mathbf{y}(x)$ changes equation (3.4) to

$$x^2 \frac{d\mathbf{y}}{dx}(x) = \hat{\mathbf{A}}(x) \mathbf{y}(x) + \hat{\mathbf{H}}(x, \mathbf{y}(x)) \quad ,$$

with

$$\hat{\mathbf{A}}(x) = \begin{pmatrix} -\lambda + x\hat{a}_1(x) & 0 \\ 0 & \lambda + x\hat{a}_2(x) \end{pmatrix}, \quad \hat{\mathbf{H}}(x, \mathbf{y}(x)) = \begin{pmatrix} \hat{H}_1(x, \mathbf{y}(x)) \\ \hat{H}_2(x, \mathbf{y}(x)) \end{pmatrix} \quad ,$$

where $\hat{a}_1, \hat{a}_2, \hat{H}_1, \hat{H}_2$ are 1-summable formal power series in x every direction $\theta \neq \arg(\pm\lambda)$.

We have

$$x^2 \frac{d\mathbf{y}}{dx}(x) = \hat{\mathbf{P}}(x)^{-1} \left(\hat{\mathbf{B}}(x) \hat{\mathbf{P}}(x) - x^2 \frac{d\hat{\mathbf{P}}}{dx}(x) \right) \mathbf{y}(x) + \hat{\mathbf{P}}(x)^{-1} \hat{\mathbf{G}}(x, \hat{\mathbf{P}}(x) \mathbf{y}(x))$$

and we want to determine $\hat{\mathbf{A}}(x)$ and $\hat{\mathbf{P}}(x)$ as above so that

$$\hat{\mathbf{B}}(x) \hat{\mathbf{P}}(x) - x^2 \frac{d\hat{\mathbf{P}}}{dx}(x) = \hat{\mathbf{A}}(x) \quad .$$

This gives four equations:

$$(3.5) \quad \begin{cases} \hat{a}_1(x) = \hat{b}_1(x) + x\hat{c}_1(x) \hat{p}_1(x) \\ \hat{a}_2(x) = \hat{b}_2(x) + x\hat{c}_2(x) \hat{p}_2(x) \\ x^2 \frac{d\hat{p}_1}{dx}(x) = (2\lambda + x\hat{b}_2(x) - x - x\hat{b}_1(x)) \hat{p}_1(x) + \hat{c}_2(x) - x^2 \hat{c}_1(x) \hat{p}_1(x)^2 \\ x^2 \frac{d\hat{p}_2}{dx}(x) = (-2\lambda + x\hat{b}_1(x) - x - x\hat{b}_2(x)) \hat{p}_2(x) + \hat{c}_1(x) - x^2 \hat{c}_2(x) \hat{p}_2(x)^2 \end{cases}.$$

Thanks to the theorem by Ramis and Sibuya on the summability of formal solutions to irregular systems [RS89], we have the existence and the 1-summability in every direction $\theta \neq \arg(\pm\lambda)$, of \hat{p}_1 and \hat{p}_2 : $(\hat{p}_1(x), \hat{p}_2(x))$ is defined as the unique formal solution to

$$\begin{cases} x^2 \frac{d\hat{p}_1}{dx}(x) = (2\lambda + x\hat{b}_2(x) - x - x\hat{b}_1(x)) \hat{p}_1(x) + \hat{c}_2(x) - x^2 \hat{c}_1(x) \hat{p}_1(x)^2 \\ x^2 \frac{d\hat{p}_2}{dx}(x) = (-2\lambda + x\hat{b}_1(x) - x - x\hat{b}_2(x)) \hat{p}_2(x) + \hat{c}_1(x) - x^2 \hat{c}_2(x) \hat{p}_2(x)^2 \end{cases}$$

such that

$$(\hat{p}_1(0), \hat{p}_2(0)) = \left(\frac{-\hat{c}_2(0)}{2\lambda}, \frac{\hat{c}_1(0)}{2\lambda} \right).$$

Notice that \hat{a}_1 and \hat{a}_2 are defined by the first two equations in (3.5). Finally, $\hat{\mathbf{H}}$ is defined by

$$\hat{\mathbf{H}}(x, \mathbf{y}) := \hat{\mathbf{P}}(x)^{-1} \hat{\mathbf{G}}(x, \hat{\mathbf{P}}(x) \mathbf{y}),$$

and it is 1-summable in every direction $\theta \neq \arg(\pm\lambda)$, by Proposition 2.20. \square

The goal of the last following lemma is to transform $\hat{a}_1(x)$ and $\hat{a}_2(x)$ in (3.3) to constant terms.

Lemma 3.13. *There exists a pair of formal power series $(\hat{q}_1, \hat{q}_2) \in (\mathbb{C}[[x]])^2$ with $\hat{q}_1(0) = \hat{q}_2(0) = 1$, which are 1-summable in every direction $\theta \neq \arg(\pm\lambda)$, such that the formal fibered diffeomorphism of the form*

$$\hat{\Phi}_3(x, y_1, y_2) = (x, \hat{q}_1(x) y_1, \hat{q}_2(x) y_2) ,$$

(which is tangent to the identity and 1-summable in every direction $\theta \neq \arg(\pm\lambda)$) conjugates \hat{Y}_2 in (3.3), to

$$\begin{aligned} \hat{Y}_3(x, \mathbf{y}) = & x^2 \frac{\partial}{\partial x} + \left((-\lambda + a_1 x) y_1 + \hat{F}_1(x, \mathbf{y}) \right) \frac{\partial}{\partial y_1} \\ & + \left((\lambda + a_2 x) y_2 + \hat{F}_2(x, \mathbf{y}) \right) \frac{\partial}{\partial y_2} , \end{aligned}$$

where \hat{F}_1, \hat{F}_2 are formal power series of order at least 2 with respect to \mathbf{y} which are 1-summable in every direction $\theta \neq \arg(\pm\lambda)$ and $(a_1, a_2) = (\hat{a}_1(0), \hat{a}_2(0))$.

Proof. We can associate to \hat{Y}_2 the following irregular differential system:

$$x^2 \frac{d\mathbf{z}}{dx}(x) = \hat{\mathbf{A}}(x) \mathbf{z}(x) + \hat{\mathbf{H}}(x, \mathbf{z}(x)) ,$$

and we are looking for a change of coordinates of the form $\mathbf{z}(x) = \hat{\mathbf{Q}}(x) \mathbf{y}(x)$, where

$$\hat{\mathbf{Q}}(x) = \begin{pmatrix} \hat{q}_1(x) & 0 \\ 0 & \hat{q}_2(x) \end{pmatrix} \in \text{GL}_2(\mathbb{C}[[x]])$$

with $\hat{q}_1(0) = \hat{q}_2(0) = 1$, such that the new system is

$$x^2 \frac{d\mathbf{y}}{dx}(x) = \mathbf{A}(x) \mathbf{y}(x) + \hat{\mathbf{F}}(x, \mathbf{y}(x)) ,$$

with

$$\mathbf{A}(x) = \begin{pmatrix} -\lambda + a_1 x & 0 \\ 0 & \lambda + a_2 x \end{pmatrix}, \quad \hat{\mathbf{F}}(x, \mathbf{y}(x)) = \begin{pmatrix} \hat{F}_1(x, \mathbf{y}(x)) \\ \hat{F}_2(x, \mathbf{y}(x)) \end{pmatrix} ,$$

and $(a_1, a_2) = (\hat{a}_1(0), \hat{a}_2(0))$.

We have

$$\begin{aligned} x^2 \frac{d\mathbf{y}}{dx}(x) = & \underbrace{\hat{\mathbf{Q}}(x)^{-1} \left(\hat{\mathbf{A}}(x) \hat{\mathbf{Q}}(x) - x^2 \frac{d\hat{\mathbf{Q}}}{dx}(x) \right)}_{=} \mathbf{y}(x) + \hat{\mathbf{Q}}(x)^{-1} \hat{\mathbf{H}}(x, \hat{\mathbf{Q}}(x) \mathbf{y}(x)) \\ & \begin{pmatrix} -\lambda + a_1 x & 0 \\ 0 & \lambda + a_2 x \end{pmatrix} \end{aligned}$$

so that

$$x^2 \frac{d\hat{\mathbf{Q}}}{dx}(x) = \hat{\mathbf{A}}(x) \hat{\mathbf{Q}}(x) - \hat{\mathbf{Q}}(x) \begin{pmatrix} -\lambda + a_1 x & 0 \\ 0 & \lambda + a_2 x \end{pmatrix}$$

and we obtain:

$$\begin{aligned}
 & \begin{cases} x^2 \frac{d\hat{q}_1}{dx}(x) = x\hat{q}_1(x) (\hat{a}_1(x) - a_1) \\ x^2 \frac{d\hat{q}_2}{dx}(x) = x\hat{q}_2(x) (\hat{a}_2(x) - a_2) \end{cases} \\
 & \iff \begin{cases} \frac{d\hat{q}_1}{dx}(x) = \hat{q}_1(x) \left(\frac{\hat{a}_1(x) - a_1}{x} \right) \\ \frac{d\hat{q}_2}{dx}(x) = \hat{q}_2(x) \left(\frac{\hat{a}_2(x) - a_2}{x} \right) \end{cases} \\
 & \iff \begin{cases} \hat{q}_1(x) = \exp \left(\int_0^x \frac{\hat{a}_1(s) - a_1}{s} ds \right) \\ \hat{q}_2(x) = \exp \left(\int_0^x \frac{\hat{a}_2(s) - a_2}{s} ds \right) \end{cases}, \text{ if we set } \hat{q}_1(0) = \hat{q}_2(0) = 1,
 \end{aligned}$$

and the expression $\int_0^x \frac{\hat{a}_j(s) - a_j}{s} ds$, for $j = 1, 2$, means the only anti-derivative of $\frac{\hat{a}_j(s) - a_j}{s}$ without constant term. Since \hat{a}_1 and \hat{a}_2 are 1-summable in every direction $\theta \neq \arg(\pm\lambda)$, the same goes for \hat{q}_1 and \hat{q}_2 , and then for \hat{F}_1 and \hat{F}_2 by Proposition 2.20. \square

3.2.2. Proof of Proposition 3.9.

We are now able to prove Proposition 3.9.

Proof of Proposition 3.9. We have to use successively Lemma 3.7 (to $Y_0 := Y_{\{x=0\}}$), followed by Proposition 3.9, then Proposition 3.14 and finally Proposition 3.18, using at each step Corollary 2.21 to obtain the 1-summability. \square

3.3. 1-summable straightening of two invariant hypersurfaces.

For any $\theta \in \mathbb{R}$, we recall that we denote by F_θ the 1-sum of a 1-summable series \hat{F} in the direction θ .

Let $\theta \in \mathbb{R}$ with $\theta \neq \arg(\pm\lambda)$ and consider a formal vector field \hat{Y} , 1-summable in the direction θ of 1-sum Y_θ , of the form

$$(3.6) \quad \hat{Y} = x^2 \frac{\partial}{\partial x} + \left(\lambda_1(x) y_1 + \hat{F}_1(x, \mathbf{y}) \right) \frac{\partial}{\partial y_1} + \left(\lambda_2(x) y_2 + \hat{F}_2(x, \mathbf{y}) \right) \frac{\partial}{\partial y_2},$$

where:

- $\lambda_1(x) = -\lambda + a_1 x$
- $\lambda_2(x) = \lambda + a_2 x$
- $\lambda \neq 0$
- $a_1, a_2 \in \mathbb{C}$
- for $j = 1, 2$,

$$\hat{F}_j(x, \mathbf{y}) = \sum_{\substack{\mathbf{n} \in \mathbb{N}^2 \\ |\mathbf{n}| \geq 2}} \hat{F}_{\mathbf{n}}^{(j)}(x) \mathbf{y}^{\mathbf{n}} \in \mathbb{C}[[x, \mathbf{y}]]$$

is 1-summable in the direction θ of 1-sum

$$F_{j,\theta}(x, \mathbf{y}) = \sum_{\substack{\mathbf{n} \in \mathbb{N}^2 \\ |\mathbf{n}| \geq 2}} F_{j,\mathbf{n},\theta}(x) \mathbf{y}^{\mathbf{n}}.$$

In particular, there exists $A, B, \mu > 0$ such that for all $\mathbf{n} \in \mathbb{N}^2$, $|n| \geq 2$, for $j = 1, 2$:

$$\forall t \in \Delta_{\theta, \epsilon, \rho}, \quad \left| \tilde{\mathcal{B}} \left(\hat{F}_{j,\mathbf{n}} \right) (t) \right| \leq A.B^{|\mathbf{n}|} \frac{\exp(\mu|t|)}{1 + \mu^2|t|^2},$$

for some $\rho > 0$ and $\epsilon > 0$ such that $(\mathbb{R}.\lambda) \cap \mathcal{A}_{\theta, \epsilon} = \emptyset$ (see Definition 2.10 and Remark 2.11 for the notations). Notice that $F_{j,\theta}$ is analytic and bounded in some sectorial neighborhood $\mathcal{S} \in \mathcal{S}_{\theta, \pi}$ of the origin. For technical reasons, we use in this subsection the alternative definition of the

Borel transform $\tilde{\mathcal{B}}$, with its associate norm $\|\cdot\|_\mu^{\text{bis}}$ (see Remarks 2.9 and 2.11 and Proposition 2.12)

Proposition 3.14. *Under the assumptions above, there exists a pair of formal power series $(\hat{\phi}_1, \hat{\phi}_2) \in (\mathbb{C}[[x, \mathbf{y}]])^2$ of order at least two with respect to \mathbf{y} which are 1-summable in every direction $\theta \neq \arg(\pm\lambda)$, such that the formal fibered diffeomorphism*

$$\hat{\Phi}(x, \mathbf{y}) = \left(x, y_1 + \hat{\phi}_1(x, \mathbf{y}), y_2 + \hat{\phi}_2(x, \mathbf{y}) \right) \quad ,$$

(which is tangent to the identity and 1-summable in every direction $\theta \neq \arg(\pm\lambda)$) conjugates \hat{Y} in (3.6) to

$$\hat{Y}_{\text{prep}} = x^2 \frac{\partial}{\partial x} + \left((-\lambda + a_1 x) + y_2 \hat{R}_1(x, \mathbf{y}) \right) y_1 \frac{\partial}{\partial y_1} + \left((\lambda + a_2 x) + y_1 \hat{R}_2(x, \mathbf{y}) \right) y_2 \frac{\partial}{\partial y_2} \quad ,$$

where $\hat{R}_1, \hat{R}_2 \in \mathbb{C}[[x, \mathbf{y}]]$ are 1-summable in every direction $\theta \neq \arg(\pm\lambda)$.

Proof. We follow and adapt the proof of analytic straightening of invariant curves for resonant saddles in two dimensions in [MM80].

We are looking for

$$\hat{\Psi}(x, \mathbf{y}) = \left(x, y_1 + \hat{\psi}_1(x, \mathbf{y}), y_2 + \hat{\psi}_2(x, \mathbf{y}) \right) \quad ,$$

with $\hat{\psi}_1, \hat{\psi}_2$ of order at least 2, and \hat{R}_1, \hat{R}_2 as above such that:

$$\hat{\Psi}_* \left(\hat{Y}_{\text{prep}} \right) = \hat{Y} \quad ,$$

i.e.

$$(3.7) \quad D\hat{\Psi} \cdot \hat{Y}_{\text{prep}} = \hat{Y} \circ \hat{\Psi} \quad .$$

Then, we will set $\Phi := \Psi^{-1}$. Let us write

$$\begin{aligned} \hat{T}_1 &:= y_1 y_2 \hat{R}_1 = \sum_{|\mathbf{n}| \geq 2} \hat{T}_{1, \mathbf{n}}(x) \mathbf{y}^{\mathbf{n}} \\ \hat{T}_2 &:= y_1 y_2 \hat{R}_2 = \sum_{|\mathbf{n}| \geq 2} \hat{T}_{2, \mathbf{n}}(x) \mathbf{y}^{\mathbf{n}} \\ \hat{\psi}_1 &= \sum_{|\mathbf{n}| \geq 2} \hat{\psi}_{1, \mathbf{n}}(x) \mathbf{y}^{\mathbf{n}} \\ \hat{\psi}_2 &= \sum_{|\mathbf{n}| \geq 2} \hat{\psi}_{2, \mathbf{n}}(x) \mathbf{y}^{\mathbf{n}} \quad , \end{aligned}$$

so that equation (3.7) becomes:

$$\begin{aligned} &x^2 \frac{\partial \hat{\psi}_1}{\partial x^2} + \left(1 + \frac{\partial \hat{\psi}_1}{\partial y_1} \right) \left(\lambda_1(x) y_1 + \hat{T}_1 \right) + \frac{\partial \hat{\psi}_1}{\partial y_2} \left(\lambda_2(x) y_2 + \hat{T}_2 \right) \\ &= \lambda_1(x) \left(y_1 + \hat{\psi}_1 \right) + \hat{F}_1 \left(x, y_1 + \hat{\psi}_1, y_2 + \hat{\psi}_2 \right) \end{aligned}$$

and

$$\begin{aligned} &x^2 \frac{\partial \hat{\psi}_2}{\partial x^2} + \frac{\partial \hat{\psi}_2}{\partial y_1} \left(\lambda_1(x) y_1 + \hat{T}_1 \right) + \left(1 + \frac{\partial \hat{\psi}_2}{\partial y_2} \right) \left(\lambda_2(x) y_2 + \hat{T}_2 \right) \\ &= \lambda_2(x) \left(y_2 + \hat{\psi}_2 \right) + \hat{F}_2 \left(x, y_1 + \hat{\psi}_1, y_2 + \hat{\psi}_2 \right) \quad . \end{aligned}$$

These equations can be written as:

$$(3.8) \quad \begin{cases} \sum_{|\mathbf{n}| \geq 2} \left(\delta_{1,\mathbf{n}}(x) \hat{\psi}_{1,\mathbf{n}}(x) + x^2 \frac{d\hat{\psi}_{1,\mathbf{n}}}{dx}(x) + \hat{T}_{1,\mathbf{n}}(x) \right) \mathbf{y}^{\mathbf{n}} \\ = \hat{F}_1(x, y_1 + \hat{\psi}_1(x, \mathbf{y}), y_2 + \hat{\psi}_2(x, \mathbf{y})) - \hat{T}_1(x) \frac{\partial \hat{\psi}_1}{\partial y_1}(x, \mathbf{y}) - \hat{T}_2(x) \frac{\partial \hat{\psi}_1}{\partial y_2}(x, \mathbf{y}) \\ =: \sum_{|\mathbf{n}| \geq 2} \zeta_{1,\mathbf{n}}(x) \mathbf{y}^{\mathbf{n}} \\ \sum_{|\mathbf{n}| \geq 2} \left(\delta_{2,\mathbf{n}}(x) \hat{\psi}_{2,\mathbf{n}}(x) + x^2 \frac{d\hat{\psi}_{2,\mathbf{n}}}{dx}(x) + \hat{T}_{2,\mathbf{n}}(x) \right) \mathbf{y}^{\mathbf{n}} \\ = \hat{F}_2(x, y_1 + \hat{\psi}_1(x, \mathbf{y}), y_2 + \hat{\psi}_2(x, \mathbf{y})) - \hat{T}_1(x) \frac{\partial \hat{\psi}_2}{\partial y_1}(x, \mathbf{y}) - \hat{T}_2(x) \frac{\partial \hat{\psi}_2}{\partial y_2}(x, \mathbf{y}) \\ =: \sum_{|\mathbf{n}| \geq 2} \zeta_{2,\mathbf{n}}(x) \mathbf{y}^{\mathbf{n}} \end{cases}$$

where $\delta_{j,\mathbf{n}}(x) = \lambda_1(x)n_1 + \lambda_2(x)n_2 - \lambda_j(x)$, $j = 1, 2$. We are looking for \hat{T}_1, \hat{T}_2 such that

$$\begin{cases} \hat{T}_{1,\mathbf{n}} = 0 & \text{, if } n_1 = 0 \text{ or } n_2 = 0 \\ \hat{T}_{2,\mathbf{n}} = 0 & \text{, if } n_1 = 0 \text{ or } n_2 = 0 \end{cases}.$$

Notice that $\zeta_{j,\mathbf{n}}$, for $j = 1, 2$ and $|\mathbf{n}| \geq 2$, depends only on the $\hat{\psi}_{i,\mathbf{k}}$'s and the $\hat{F}_{i,\mathbf{k}}$'s, for $i = 1, 2$, $|\mathbf{k}| < \mathbf{n}$. We can then determine the coefficients $\hat{\psi}_{j,\mathbf{n}}$ and $\hat{T}_{j,\mathbf{n}}$, $j = 1, 2$, $|\mathbf{n}| \geq 2$, by induction on $|\mathbf{n}|$, setting

$$\begin{cases} \hat{T}_{1,\mathbf{n}} = 0 & \text{, if } n_1 = 0 \text{ or } n_2 = 0 \\ \hat{T}_{2,\mathbf{n}} = 0 & \text{, if } n_1 = 0 \text{ or } n_2 = 0 \\ \hat{\psi}_{1,\mathbf{n}} = 0 & \text{, if } n_1 \geq 1 \text{ and } n_2 \geq 1 \\ \hat{\psi}_{2,\mathbf{n}} = 0 & \text{, if } n_1 \geq 1 \text{ and } n_2 \geq 1 \end{cases},$$

and solving for each $\mathbf{n} = (n_1, n_2) \in \mathbb{N}^2$ with $|\mathbf{n}| \geq 2$, the equations

$$\begin{cases} \delta_{1,\mathbf{n}}(x) \hat{\psi}_{1,\mathbf{n}}(x) + x^2 \frac{d\hat{\psi}_{1,\mathbf{n}}}{dx}(x) = \zeta_{1,\mathbf{n}}(x) & \text{, if } n_1 = 0 \text{ or } n_2 = 0 \\ \delta_{2,\mathbf{n}}(x) \hat{\psi}_{2,\mathbf{n}}(x) + x^2 \frac{d\hat{\psi}_{2,\mathbf{n}}}{dx}(x) = \zeta_{2,\mathbf{n}}(x) & \text{, if } n_1 = 0 \text{ or } n_2 = 0 \end{cases}.$$

Lemma 3.15. *There exists $\beta > 4\pi$, $M > 0$ such that for all $\mathbf{n} \in \mathbb{N}^2$ with $|\mathbf{n}| \geq 2$, and for $j = 1, 2$, $\|\zeta_{j,\mathbf{n}}\|_{\beta}^{\text{bis}} < +\infty$ and:*

$$\left\| \hat{\psi}_{j,\mathbf{n}} \right\|_{\beta}^{\text{bis}} \leq M \cdot \|\zeta_{j,\mathbf{n}}\|_{\beta}^{\text{bis}},$$

where the norm corresponds to the domain $\Delta_{\theta,\epsilon,\rho}$ (see Definition 2.10).

Proof. For $\mathbf{n} = (n_1, n_2) \in \mathbb{N}^2$ with $n_1 + n_2 \geq 2$ we want to solve:

$$\begin{cases} \delta_{1,\mathbf{n}}(x) = \lambda_1(x)(n_1 - 1) + \lambda_2(x)n_2 = \begin{cases} \lambda(n_2 + 1) + x(-a_1 + a_2n_2) & \text{, if } n_1 = 0 \\ -\lambda(n_1 - 1) + a_1x(n_1 - 1) & \text{, if } n_2 = 0 \end{cases} \\ \delta_{2,\mathbf{n}}(x) = \lambda_2(x)(n_2 - 1) + \lambda_1(x)n_1 = \begin{cases} \lambda(n_2 - 1) + a_2x(n_2 - 1) & \text{, if } n_1 = 0 \\ -\lambda(n_1 + 1) + x(-a_2 + a_1n_1) & \text{, if } n_2 = 0 \end{cases} \end{cases}.$$

We will only deal with $\delta_{1,\mathbf{n}}(x)$ (the case of $\delta_{2,\mathbf{n}}(x)$ being similar). Notice that we are exactly in the situation of Proposition 2.30. In particular, using notation in this definition, we respectively have:

$$\begin{cases} k = \lambda(n_2 + 1), \alpha = \frac{(-a_1 + a_2 n_2)}{\lambda(n_2 + 1)}, \\ d_k = \min \{ |\lambda(n_2 + 1)| - \rho, |\lambda(n_2 + 1)| |\sin(\theta + \epsilon)|, |\lambda(n_2 + 1)| |\sin(\theta - \epsilon)| \} \end{cases}$$

(when $n_1 = 0$)

and

$$\begin{cases} k = -\lambda(n_1 + 1), \alpha = \frac{(-a_2 + a_1 n_1)}{-\lambda(n_1 + 1)}, \\ d_k = \min \{ |\lambda(n_1 + 1)| - \rho, |\lambda(n_1 + 1)| |\sin(\theta + \epsilon)|, |\lambda(n_1 + 1)| |\sin(\theta - \epsilon)| \} \end{cases}$$

(when $n_2 = 0$).

We can chose the domain $\Delta_{\theta, \epsilon, \rho}$ corresponding to the 1-summability of \hat{F}_1 and \hat{F}_2 with $0 < \rho < |\lambda|$, so that $d_k > 0$, since $\epsilon > 0$ is such that $(\mathbb{R} \cdot \lambda) \cap \mathcal{A}_{\theta, \epsilon} = \emptyset$. Finally, we chose

$$\beta > \frac{C(|a_1| + |a_2|)}{\min \{ |\lambda| - \rho, |\lambda \sin(\theta + \epsilon)|, |\lambda \sin(\theta - \epsilon)| \}} > 0,$$

(with $C = \frac{2 \exp(2)}{5} + 5$), so that $\left\| \hat{F}_1 \right\|_{\beta}^{\text{bis}} < +\infty$. This choice of β implies $\beta d_k > C|\alpha k|$ as needed in Proposition 2.30, in both considered situations, namely $n_1 = 0$ and $n_2 = 0$ respectively. Since for $j = 1, 2$ and $|\mathbf{n}| \geq 2$, $\zeta_{j, \mathbf{n}}$ depends only on the $\hat{\psi}_{i, \mathbf{k}}$'s and the $\hat{F}_{i, \mathbf{k}}$'s, for $i = 1, 2$, $|\mathbf{k}| < \mathbf{n}$, we deduce by induction that

$$\begin{cases} \|\zeta_{1, \mathbf{n}}\|_{\beta}^{\text{bis}} < +\infty & , \text{ if } n_1 = 0 \text{ or } n_2 = 0 \\ \|\zeta_{2, \mathbf{n}}\|_{\beta}^{\text{bis}} < +\infty & , \text{ if } n_1 = 0 \text{ or } n_2 = 0 \end{cases}$$

and then, thanks to Proposition 2.30:

$$\left\| \hat{\psi}_{j, \mathbf{n}} \right\|_{\beta}^{\text{bis}} \leq \left(\frac{\beta}{\beta(|\lambda| - \rho) - C(|a_1| + |a_2|)} \right) \cdot \|\zeta_{j, \mathbf{n}}\|_{\beta}^{\text{bis}}, \text{ for } j = 1, 2.$$

The lemma is proved, with

$$M = \left(\frac{\beta}{\beta \min \{ |\lambda| - \rho, |\lambda \sin(\theta + \epsilon)|, |\lambda \sin(\theta - \epsilon)| \} - C(|a_1| + |a_2|)} \right).$$

□

In order to finish the proof of Proposition 3.14, we have to prove that for $j = 1, 2$, the series $\overline{\hat{\psi}_j} := \sum_{\mathbf{n} \in \mathbb{N}^2} \left\| \hat{\psi}_{j, \mathbf{n}} \right\|_{\beta}^{\text{bis}} \mathbf{y}^{\mathbf{n}}$ is convergent in a poly-disc $\mathbf{D}(\mathbf{0}, \mathbf{r})$, with $\mathbf{r} = (r_1, r_2) \in (\mathbb{R}_{>0})^2$ (then, Corollary 2.28 gives 1-summability). We will prove this by using a method of dominant series. Let us introduce some useful notations. If $(\mathfrak{B}, \|\cdot\|)$ is a Banach algebra, for any formal power series $f(\mathbf{y}) = \sum_{\mathbf{n}} f_{\mathbf{n}} \mathbf{y}^{\mathbf{n}} \in \mathfrak{B}[[\mathbf{y}]]$, we define $\overline{f} := \sum_{\mathbf{n}} \|f_{\mathbf{n}}\| \mathbf{y}^{\mathbf{n}}$, and $\overline{\overline{f}}(y) := \overline{f}(y, y)$. If $g = \sum_{\mathbf{n}} g_{\mathbf{n}} \mathbf{y}^{\mathbf{n}} \in \mathfrak{B}[[\mathbf{y}]]$ is another formal power series, we write $\overline{f} \prec \overline{g}$ if for all $\mathbf{n} \in \mathbb{N}^2$, we have $\|f_{\mathbf{n}}\| \leq \|g_{\mathbf{n}}\|$. We remind the following classical result (the proof is performed in [RR11] when $(\mathfrak{B}, \|\cdot\|) = (\mathbb{C}, |\cdot|)$, but the same proof works for any Banach algebra).

Lemma 3.16. [RR11, Theorem 2.2 p.48] *For $j = 1, 2$, let $f_j = \sum_{|\mathbf{n}| \geq 2} f_{j, \mathbf{n}} \mathbf{y}^{\mathbf{n}} \in \mathfrak{B}[[\mathbf{y}]]$ be two formal power series with coefficients in a Banach algebra $(\mathfrak{B}, \|\cdot\|)$, and of order at least two. Consider also*

two other series $g_j = \sum_{|\mathbf{n}| \geq 2} g_{j,\mathbf{n}} \mathbf{y}^{\mathbf{n}} \in \mathfrak{B}\{\mathbf{y}\}$, $j = 1, 2$, of order at least two, which have a non-zero radius of convergence at the origin. Assume that there exists $\sigma > 0$ such that for $j = 1, 2$:

$$\sigma \overline{f_j} \prec \overline{g_j} (y_1 + \overline{f_1}, y_2 + \overline{f_2}) \quad .$$

Then, f_1 and f_2 have a non-zero radius of convergence.

Taking $\beta > 4\pi$, according to Proposition 2.12, for all $\hat{f}, \hat{g} \in \mathfrak{B}_\beta^{\text{bis}}$, we have:

$$\left\| \hat{f} \hat{g} \right\|_\beta^{\text{bis}} \leq \left\| \hat{f} \right\|_\beta^{\text{bis}} \left\| \hat{g} \right\|_\beta^{\text{bis}} \quad .$$

This implies that $(\mathfrak{B}_\beta^{\text{bis}}, \|\cdot\|_\beta^{\text{bis}})$ is a Banach algebra as needed in the above lemma. It remains to prove that there exists $\sigma > 0$ such that for $j = 1, 2$:

$$\sigma \overline{\psi_j} \prec \overline{F_j} (y_1 + \overline{\psi_1}, y_2 + \overline{\psi_2}) \quad .$$

Remember that there exists $M > 0$ such that for $j = 1, 2$:

$$\left\| \hat{\psi}_{j,\mathbf{n}} \right\|_\beta^{\text{bis}} \leq M \cdot \left\| \zeta_{j,\mathbf{n}} \right\|_\beta^{\text{bis}}$$

where

$$\begin{cases} \zeta_1 := \sum_{|\mathbf{n}| \geq 2} \zeta_{1,\mathbf{n}}(x) \mathbf{y}^{\mathbf{n}} \\ = \hat{F}_1(x, y_1 + \hat{\psi}_1(x, \mathbf{y}), y_2 + \hat{\psi}_2(x, \mathbf{y})) - \hat{T}_1(x) \frac{\partial \hat{\psi}_1}{\partial y_1}(x, \mathbf{y}) - \hat{T}_2(x) \frac{\partial \hat{\psi}_1}{\partial y_2}(x, \mathbf{y}) \\ \zeta_2 := \sum_{|\mathbf{n}| \geq 2} \zeta_{2,\mathbf{n}}(x) \mathbf{y}^{\mathbf{n}} \\ = \hat{F}_2(x, y_1 + \hat{\psi}_1(x, \mathbf{y}), y_2 + \hat{\psi}_2(x, \mathbf{y})) - \hat{T}_1(x) \frac{\partial \hat{\psi}_2}{\partial y_1}(x, \mathbf{y}) - \hat{T}_2(x) \frac{\partial \hat{\psi}_2}{\partial y_2}(x, \mathbf{y}) \end{cases} \quad .$$

If we set $\sigma := \frac{1}{M}$, then we have

$$\begin{cases} \sigma \overline{\hat{\psi}_1} \prec \overline{\zeta_1} \prec \overline{\hat{F}_1} (x, y_1 + \overline{\hat{\psi}_1}(x, \mathbf{y}), y_2 + \overline{\hat{\psi}_2}(x, \mathbf{y})) + \overline{\hat{T}_1}(x) \frac{\partial \overline{\hat{\psi}_1}}{\partial y_1}(x, \mathbf{y}) + \overline{\hat{T}_2}(x) \frac{\partial \overline{\hat{\psi}_1}}{\partial y_2}(x, \mathbf{y}) \\ \sigma \overline{\hat{\psi}_2} \prec \overline{\zeta_2} \prec \overline{\hat{F}_2} (x, y_1 + \overline{\hat{\psi}_1}(x, \mathbf{y}), y_2 + \overline{\hat{\psi}_2}(x, \mathbf{y})) + \overline{\hat{T}_1}(x) \frac{\partial \overline{\hat{\psi}_2}}{\partial y_1}(x, \mathbf{y}) + \overline{\hat{T}_2}(x) \frac{\partial \overline{\hat{\psi}_2}}{\partial y_2}(x, \mathbf{y}) \end{cases} \quad .$$

Moreover, we recall that

$$\begin{cases} \hat{T}_{1,\mathbf{n}} = 0 & , \text{ if } n_1 = 0 \text{ or } n_2 = 0 \\ \hat{T}_{2,\mathbf{n}} = 0 & , \text{ if } n_1 = 0 \text{ or } n_2 = 0 \\ \hat{\psi}_{1,\mathbf{n}} = 0 & , \text{ if } n_1 \geq 1 \text{ and } n_2 \geq 1 \\ \hat{\psi}_{2,\mathbf{n}} = 0 & , \text{ if } n_1 \geq 1 \text{ and } n_2 \geq 1 \end{cases} \quad ,$$

so that we have in fact more precise dominant relations:

$$\begin{cases} \sigma \overline{\hat{\psi}_1} \prec \overline{\zeta_1} \prec \overline{\hat{F}_1} (x, y_1 + \overline{\hat{\psi}_1}(x, \mathbf{y}), y_2 + \overline{\hat{\psi}_2}(x, \mathbf{y})) \\ \sigma \overline{\hat{\psi}_2} \prec \overline{\zeta_2} \prec \overline{\hat{F}_2} (x, y_1 + \overline{\hat{\psi}_1}(x, \mathbf{y}), y_2 + \overline{\hat{\psi}_2}(x, \mathbf{y})) \end{cases} \quad .$$

It remains to apply the lemma above to conclude. □

Remark 3.17. In the previous proposition, assume that for $j = 1, 2$,

$$\hat{F}_j(x, \mathbf{y}) = \sum_{\mathbf{n} \in \mathbb{N}^2, |\mathbf{n}| \geq 2} \hat{F}_{\mathbf{n}}^{(j)}(x) \mathbf{y}^{\mathbf{n}}$$

in the expression of \hat{Y} satisfies

$$\begin{cases} \hat{F}_{\mathbf{n}}^{(1)}(0) = 0 & , \forall \mathbf{n} = (n_1, n_2) \mid n_1 + n_2 \geq 2 \text{ and } (n_1 = 0 \text{ or } n_2 = 0) \\ \hat{F}_{\mathbf{n}}^{(2)}(0) = 0 & , \forall \mathbf{n} = (n_1, n_2) \mid n_1 + n_2 \geq 2 \text{ and } (n_1 = 0 \text{ or } n_2 = 0) \end{cases}.$$

Then, the diffeomorphism $\hat{\Phi}$ in the proposition can be chosen to be the identity on $\{x = 0\}$, so that

$$\begin{cases} y_1 y_2 \hat{R}_1(x, \mathbf{y}) &= \hat{F}_1(0, \mathbf{y}) + x \hat{S}_1(x, \mathbf{y}) \\ y_1 y_2 \hat{R}_2(x, \mathbf{y}) &= \hat{F}_2(0, \mathbf{y}) + x \hat{S}_2(x, \mathbf{y}) \end{cases},$$

where \hat{S}_1, \hat{S}_2 are 1-summable in the direction $\theta \neq \arg(\pm\lambda)$ and $\hat{F}_1(0, \mathbf{y}), \hat{F}_2(0, \mathbf{y}) \in \mathbb{C}\{\mathbf{y}\}$ are convergent in neighborhood of the origin in \mathbb{C}^2 . Indeed, we easily see by induction on $|\mathbf{n}| = n_1 + n_2 \geq 2$ that $\hat{\psi}_1$ and $\hat{\psi}_2$ can be chosen “divisible” by x , and that ζ_1, ζ_2 are such that $\zeta_{j,\mathbf{n}}(x)$ is also “divisible” by x if $n_1 = 0$ or $n_2 = 0$.

3.4. 1-summable normal form up to arbitrary order N .

We consider now a (formal) non-degenerate diagonal doubly-resonant saddle node, which is supposed to be div-integrable and 1-summable in every direction $\theta \neq \arg(\pm\lambda)$, of the form

$$\begin{aligned} \hat{Y}_{\text{prep}} &= x^2 \frac{\partial}{\partial x} + \left(-\lambda + a_1 x - d(y_1 y_2) + x \hat{S}_1(x, \mathbf{y}) \right) y_1 \frac{\partial}{\partial y_1} \\ &\quad + \left(\lambda + a_2 x + d(y_1 y_2) + x \hat{S}_2(x, \mathbf{y}) \right) y_2 \frac{\partial}{\partial y_2}, \end{aligned}$$

where:

- $\lambda \in \mathbb{C} \setminus \{0\}$;
- $\hat{S}_1, \hat{S}_2 \in \mathbb{C}[[x, \mathbf{y}]]$ are of order at least one with respect to \mathbf{y} and 1-summable in every direction $\theta \in \mathbb{R}$ with $\theta \neq \arg(\pm\lambda)$;
- $a := \text{res}(\hat{Y}_{\text{prep}}) = a_1 + a_2 \notin \mathbb{Q}_{\leq 0}$;
- $d(v) \in v\mathbb{C}\{v\}$ is the germ of an analytic function in $v := y_1 y_2$ vanishing at the origin.

As usual, we denote by $Y_{\text{prep},\theta}, S_{1,\theta}, S_{2,\theta}$ the respective 1-sums of $\hat{Y}, \hat{S}_1, \hat{S}_2$ in the direction θ . Let us introduce some useful notations:

$$\hat{Y}_{\text{prep}} = Y_0 + D \vec{\mathcal{C}} + R \vec{\mathcal{R}},$$

where

- $\vec{\mathcal{C}} := -y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2}$
- $\vec{\mathcal{R}} := y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2}$
- $Y_0 := \lambda \vec{\mathcal{C}} + x \left(x \frac{\partial}{\partial x} + a_1 y_1 \frac{\partial}{\partial y_1} + a_2 y_2 \frac{\partial}{\partial y_2} \right)$
- $D(x, \mathbf{y}) = d(y_1 y_2) + x D^{(1)}(x, \mathbf{y}) = d(y_1 y_2) + x \left(\frac{\hat{S}_2 - \hat{S}_1}{2} \right)$ is 1-summable in the direction θ of 1-sum D_θ : it is called the “*tangential*” part. D_θ is also dominated by $\|\mathbf{y}\| = \max(|y_1|, |y_2|)$ (D is of order at least one with respect to \mathbf{y}).
- $R(x, \mathbf{y}) = x R^{(1)}(x, \mathbf{y}) = x \left(\frac{\hat{S}_2 + \hat{S}_1}{2} \right)$ is 1-summable in the direction θ of 1-sum R_θ : it is called the “*radial*” part. R_θ is also dominated by $\|\mathbf{y}\|_\infty = \max(|y_1|, |y_2|)$ (R is of order at least one with respect to \mathbf{y}).

The following proposition gives the existence of a 1-summable normalizing map, up to any order $N \in \mathbb{N}_{>0}$, with respect to x .

Proposition 3.18. *Let*

$$\hat{Y}_{\text{prep}} = Y_0 + D \vec{\mathcal{C}} + R \vec{\mathcal{R}}$$

be as above.

Then for all $N \in \mathbb{N}_{>0}$ there exist $d^{(N)}(v) \in \mathbb{C}\{v\}$ of order at least one and $\Phi^{(N)} \in \text{Diff}_{\text{fib}}(\mathbb{C}^3, 0, \text{Id})$ which conjugates \hat{Y}_{prep} (resp. its 1-sums $Y_{\text{prep}, \theta}$ in the direction θ) to

$$\begin{aligned} Y^{(N)} &= Y_0 + \left(d^{(N)}(y_1 y_2) + x^N D^{(N)}(x, \mathbf{y}) \right) \vec{\mathcal{C}} + x^N R^{(N)}(x, \mathbf{y}) \vec{\mathcal{R}} \\ \left(\text{resp. } Y_{\theta}^{(N)} \right) &= Y_0 + \left(d^{(N)}(y_1 y_2) + x^N D_{\pm}^{(N)}(x, \mathbf{y}) \right) \vec{\mathcal{C}} + x^N R_{\theta}^{(N)}(x, \mathbf{y}) \vec{\mathcal{R}} \end{aligned} \quad ,$$

where $D^{(N)}, R^{(N)}$ are 1-summable in the direction θ , of order at least one with respect to \mathbf{y} , of 1-sums $D_{\theta}^{(N)}, R_{\theta}^{(N)}$ in the direction θ . Moreover, one can choose $d^{(2)} = \dots = d^{(N)}$ for all $N \geq 2$, and $d^{(1)} = d$.

Proof. The proof is performed by induction on N .

- The case $N = 1$ is the initial situation here, and is already proved with $\hat{Y}_{\text{prep}} = Y^{(1)}$.
 - Assume that the result holds for $N \in \mathbb{N}_{>0}$.
- (1) We start with the radial part. Let us write

$$R^{(N)}(x, \mathbf{y}) = \sum_{n_1 + n_2 \geq 1} R_{n_1, n_2}^{(N)}(x) y_1^{n_1} y_2^{n_2}$$

and

$$R_{\text{res}}^{(N)}(0, v) = \sum_{k \geq 1} R_{k, k}^{(N)}(0) v^k \quad .$$

We are looking for an analytic solution τ to the equations:

$$\begin{aligned} \mathcal{L}_{Y^{(N)}}(\tau) &= -x^N R^{(N)} + \left(x^{N+1} \tilde{R}^{(N+1)} \right) \circ \Lambda_{\tau} \\ \mathcal{L}_{Y_{\theta}^{(N)}}(\tau) &= -x^N R_{\theta}^{(N)} + \left(x^{N+1} \tilde{R}_{\theta}^{(N+1)} \right) \circ \Lambda_{\tau} \end{aligned} \quad ,$$

for a convenient choice of $\tilde{R}^{(N+1)}, \tilde{R}_{\theta}^{(N+1)}$, with

$$\Lambda_{\tau}(x, \mathbf{y}) := (x, y_1 \exp(\tau(x, \mathbf{y})), y_2 \exp(\tau(x, \mathbf{y}))) \quad ,$$

and

$$\tau(x, \mathbf{y}) = x^{N-1} \tau_0(y_1 y_2) + x^N \tau_1(\mathbf{y}) \quad ,$$

where $\tau_1(\mathbf{y}) = \sum_{j_1 \neq j_2} \tau_{1, j_1 j_2} y_1^{j_1} y_2^{j_2}$. More concretely, Λ_{τ} is the formal flow of $\vec{\mathcal{R}}$ at “time”

$\tau(x, \mathbf{y})$.

If we admit for a moment that such an analytic solution τ exists, then $\Lambda_{\tau} \in \text{Diff}_{\text{fib}}(\mathbb{C}^3, 0; \text{Id})$ and therefore $\Lambda_{\tau}^{-1} \in \text{Diff}_{\text{fib}}(\mathbb{C}^3, 0; \text{Id})$. If we consider $d^{(N)}$ and $\tilde{D}^{(N)}$ such that

$$\begin{aligned} d^{(N+1)}(z_1 z_2) + x^N \tilde{D}^{(N)}(x, \mathbf{z}) &:= \left(d^{(N)}(y_1 y_2) + x^N D^{(N)}(x, \mathbf{y}) \right) \circ \Lambda_{\tau}^{-1}(x, \mathbf{z}) \\ d^{(N+1)}(z_1 z_2) + x^N \tilde{D}_{\theta}^{(N)}(x, \mathbf{z}) &:= \left(d^{(N)}(y_1 y_2) + x^N D_{\theta}^{(N)}(x, \mathbf{y}) \right) \circ \Lambda_{\tau}^{-1}(x, \mathbf{z}) \end{aligned}$$

then the two equations given in (3.9) imply that

$$\begin{aligned} (\Lambda_{\tau})_* \left(Y^{(N)} \right) &= Y_0 + \left(d^{(N+1)}(z_1 z_2) + x^N \tilde{D}^{(N)}(x, \mathbf{z}) \right) \vec{\mathcal{C}} \\ &\quad + x^{N+1} \tilde{R}^{(N+1)}(x, \mathbf{z}) \vec{\mathcal{R}} \\ (\Lambda_{\tau})_* \left(Y_{\theta}^{(N)} \right) &= Y_0 + \left(d^{(N+1)}(z_1 z_2) + x^N \tilde{D}_{\theta}^{(N)}(x, \mathbf{z}) \right) \vec{\mathcal{C}} \\ &\quad + x^{N+1} \tilde{R}_{\theta}^{(N+1)}(x, \mathbf{z}) \vec{\mathcal{R}} \quad . \end{aligned}$$

Indeed:

$$\begin{aligned}
 D\Lambda_\tau \cdot Y^{(N)} &= \begin{pmatrix} \mathcal{L}_{Y^{(N)}}(x) \\ \mathcal{L}_{Y^{(N)}}(y_1 \exp(\tau(x, \mathbf{y}))) \\ \mathcal{L}_{Y^{(N)}}(y_2 \exp(\tau(x, \mathbf{y}))) \end{pmatrix} \\
 &= \begin{pmatrix} x^2 \\ (\mathcal{L}_{Y^{(N)}}(y_1) + y_1(\mathcal{L}_{Y^{(N)}}(\tau))) \exp(\tau(x, \mathbf{y})) \\ (\mathcal{L}_{Y^{(N)}}(y_2) + y_2(\mathcal{L}_{Y^{(N)}}(\tau))) \exp(\tau(x, \mathbf{y})) \end{pmatrix} \\
 &= \left(Y_0 + \left(d^{(N+1)} + x^N \tilde{D}^{(N)} \right) \vec{\mathcal{C}} + x^{N+1} \tilde{R}^{(N+1)} \vec{\mathcal{R}} \right) \circ \Lambda_\tau(x, \mathbf{y}) .
 \end{aligned}$$

These computations are also true with the corresponding 1-sums of formal objects considered here, *i.e.* with $Y_\theta^{(N)}, D_\theta^{(N)}, \tilde{D}_\theta^{(N)}, \tilde{R}_\theta^{(N+1)}$ instead of $Y^{(N)}, D^{(N)}, \tilde{D}^{(N)}, \tilde{R}^{(N+1)}$ respectively. We use Proposition 2.20 to obtain the 1-summability of the objects defined by compositions.

Let us prove that there exists a germ of analytic function of the form

$$\tau(x, \mathbf{y}) = x^{N-1} \tau_0(y_1 y_2) + x^N \tau_1(\mathbf{y}) \quad ,$$

of order at least one with respect to \mathbf{y} in the origin, with

$$\tau_1(\mathbf{y}) = \sum_{j_1 \neq j_2} \tau_{1, j_1 j_2} y_1^{j_1} y_2^{j_2}$$

satisfying equation (3.9). This equation can be written

$$\begin{aligned}
 &x^2 \frac{\partial \tau}{\partial x} + \left(-\lambda + a_1 x - d^{(N)}(y_1 y_2) - x^N D^{(N)}(x, \mathbf{y}) + x^N R^{(N)}(x, \mathbf{y}) \right) y_1 \frac{\partial \tau}{\partial y_1} \\
 &+ \left(\lambda + a_2 x + d^{(N)}(y_1 y_2) + x^N D^{(N)}(x, \mathbf{y}) + x^N R^{(N)}(x, \mathbf{y}) \right) y_2 \frac{\partial \tau}{\partial y_2} \\
 &= -x^N R^{(N)} + \left(x^{N+1} \tilde{R}^{(N+1)} \right) \circ \Lambda_\tau \quad ,
 \end{aligned}$$

or equivalently

$$\begin{aligned}
 &x^2 \frac{\partial \tau}{\partial x} + a_1 x y_1 \frac{\partial \tau}{\partial y_1} + a_2 x y_2 \frac{\partial \tau}{\partial y_2} + \left(\lambda + d^{(N)}(y_1 y_2) + x^N D^{(N)}(x, \mathbf{y}) \right) \mathcal{L}_{\vec{\mathcal{C}}}(\tau) \\
 &+ \left(x^N R^{(N)}(x, \mathbf{y}) \right) \mathcal{L}_{\vec{\mathcal{R}}}(\tau) = -x^N R^{(N)} + \left(x^{N+1} \tilde{R}^{(N+1)} \right) \circ \Lambda_\tau \quad .
 \end{aligned}$$

Let us consider terms of degree N with respect to x :

$$\begin{aligned}
 (N-1) \tau_0(y_1 y_2) + (a_1 + a_2 + 2\delta_{N,1} R^{(N)}(0, \mathbf{y})) y_1 y_2 \frac{\partial \tau_0}{\partial v}(y_1 y_2) \\
 + (\lambda + d^{(N)}(y_1 y_2)) \mathcal{L}_{\vec{\mathcal{C}}}(\tau_1) = -R^{(N)}(0, \mathbf{y})
 \end{aligned} \tag{3.10}$$

(here $\delta_{N,1}$ is the Kronecker notation). We use now the fact that $\text{Im}(\mathcal{L}_{\vec{\mathcal{C}}}) \oplus \text{Ker}(\mathcal{L}_{\vec{\mathcal{C}}})$ is a direct sum, and that $\text{Ker}(\mathcal{L}_{\vec{\mathcal{C}}})$ is the set of formal power series in the resonant monomial $v = y_1 y_2$. Isolating the term $\mathcal{L}_{\vec{\mathcal{C}}}(\tau_1)$ on the one hand, and the others on the other hand, the direct sum above gives us:

$$\begin{cases} v \left(a_1 + a_2 + 2\delta_{N,1} R_{\text{res}}^{(N)}(0, v) \right) \frac{d\tau_0}{dv}(v) + (N-1) \tau_0(v) = -R_{\text{res}}^{(N)}(0, v) \\ \tau_0(0) = 0 \end{cases}$$

and

$$\begin{cases} \mathcal{L}_{\vec{c}}(\tau_1) = \frac{-1}{\lambda + d^{(N)}(y_1 y_2)} \left(\left(2\delta_{N,1} \left(R^{(N)}(0, \mathbf{y}) - R_{\text{res}}^{(N)}(0, v) \right) \right) y_1 y_2 \frac{d\tau_0}{dv}(y_1 y_2) \right. \\ \quad \left. + R^{(N)}(0, \mathbf{y}) - R_{\text{res}}^{(N)}(0, v) \right) \\ \tau_1(0) = 0. \end{cases}$$

Since $R^{(N)}$ is analytic with respect to \mathbf{y} , $R_{\text{res}}^{(N)}(0, v)$ is analytic near $v = 0$. Furthermore, as $R_{\text{res}}^{(N)}(0, 0) = 0$ and $a_1 + a_2 \notin \mathbb{Q}_{\leq 0}$, the first of the two equation above has a unique formal solution τ_0 with $\tau_0(0)$, and this solution is convergent in a neighborhood of the origin. Once τ_0 is determined, there exists a unique formal solution τ_1 to the second equation satisfying $\tau_1(\mathbf{y}) = \sum_{j_1 \neq j_2} \tau_{1,j_1 j_2} y_1^{j_1} y_2^{j_2}$, which is moreover convergent in a neighborhood of the origin of \mathbb{C}^2 .

Therefore Λ_τ is a germ of analytic diffeomorphism fixing the origin, fibered, tangent to the identity and conjugates $Y^{(N)}$ (*resp.* $Y_\theta^{(N)}$) to $\tilde{Y}^{(N)} := (\Lambda_\tau)_* \left(Y^{(N)} \right)$ (*resp.* $\tilde{Y}_\theta^{(N)} := (\Lambda_\tau)_* \left(Y_\theta^{(N)} \right)$).

Equation (3.10) implies that $\left(\mathcal{L}_{Y^{(N)}}(\tau) + x^N R^{(N)} \right)$ and $\left(\mathcal{L}_{Y_\theta^{(N)}}(\tau) + x^N R_\theta^{(N)} \right)$ are divisible by x^{N+1} , so that we can define:

$$\begin{aligned} \tilde{R}^{(N+1)}(x, \mathbf{z}) &:= \left(\frac{\mathcal{L}_{Y^{(N)}}(\tau) + x^N R^{(N)}}{x^{N+1}} \right) \circ \Lambda_\tau^{-1}(x, \mathbf{z}) \\ \tilde{R}_\theta^{(N+1)}(x, \mathbf{z}) &:= \left(\frac{\mathcal{L}_{Y_\theta^{(N)}}(\tau) + x^N R_\theta^{(N)}}{x^{N+1}} \right) \circ \Lambda_\tau^{-1}(x, \mathbf{z}) \quad . \end{aligned}$$

By Proposition 2.20, $\tilde{R}^{(N+1)}$ (*resp.* $\tilde{D}^{(N)}$) is 1-summable in the direction θ , of 1-sum $\tilde{R}_\theta^{(N+1)}$ (*resp.* $\tilde{D}_\theta^{(N)}$).

Finally, notice that $d^{(N+1)} \circ \Lambda_\tau(0, \mathbf{y}) = d^{(N)}(y_1, y_2)$, $\tau(0, \mathbf{y}) = 0$ and then $\Lambda_\tau(0, \mathbf{y}) = (0, y_1, y_2)$ if $N > 1$, so that $d^{(N+1)} = d^{(N)}$ when $N > 1$.

(2) No we deal with the tangential part. Let us write

$$\tilde{D}^{(N)}(x, \mathbf{z}) = \sum_{n_1 + n_2 \geq 1} \tilde{D}_{n_1, n_2}^{(N)}(x) z_1^{n_1} z_2^{n_2}$$

and

$$\tilde{D}_{\text{res}}^{(N)}(0, v) = \sum_{k \geq 1} \tilde{D}_{k, k}^{(N)}(0) v^k \quad .$$

Exactly as in the previous case which dealt with the “radial part” (in fact the computations are even easier here), we can prove the existence of a germ of an analytic function σ , solution to the equation:

$$\begin{aligned} \mathcal{L}_{\tilde{Y}^{(N)}}(\sigma) &= -x^N \tilde{D}^{(N)} + \left(x^{N+1} D^{(N+1)} \right) \circ \Gamma_\sigma \\ \mathcal{L}_{\tilde{Y}_\theta^{(N)}}(\sigma) &= -x^N \tilde{D}_\theta^{(N)} + \left(x^{N+1} D_\theta^{(N+1)} \right) \circ \Gamma_\sigma \quad , \end{aligned} \tag{3.11}$$

for a good choice of $D^{(N+1)}, D_\theta^{(N+1)}$, with

$$\Gamma_\sigma(x, \mathbf{z}) := (x, y_1 \exp(-\sigma(x, \mathbf{z})), y_2 \exp(\sigma(x, \mathbf{z})))$$

and

$$\sigma(x, \mathbf{z}) = x^{N-1} \sigma_0(z_1 z_2) + x^N \sigma_1(\mathbf{z}) \quad ,$$

where $\sigma_1(\mathbf{z}) = \sum_{j_1 \neq j_2} \sigma_{1,j_1,j_2} z_1^{j_1} z_2^{j_2}$. Notice that Γ_σ is the formal flow of $\vec{\mathcal{C}}$ at “time” $\sigma(x, \mathbf{z})$.

Again, as in the first case with the “radial part”, we have on a $\Gamma_\sigma \in \text{Diff}_{\text{fib}}(\mathbb{C}^3, 0; \text{Id})$ and then also $\Gamma_\sigma^{-1} \in \text{Diff}_{\text{fib}}(\mathbb{C}^3, 0; \text{Id})$. If we consider $R^{(N+1)}$ and $R_\theta^{(N+1)}$ such that

$$\begin{aligned} R^{(N+1)}(x, \mathbf{y}) &:= \tilde{R}^{(N+1)} \circ \Gamma_\sigma^{-1}(x, \mathbf{y}) \\ R_\theta^{(N+1)}(x, \mathbf{z}) &:= \tilde{R}_\theta^{(N+1)} \circ \Gamma_\sigma^{-1}(x, \mathbf{y}) \end{aligned} \quad ,$$

then it follows from (3.11) that

$$\begin{aligned} (\Gamma_\sigma)_* \left(\tilde{Y}^{(N)} \right) &= Y_0 + \left(d^{(N+1)}(y_1 y_2) + x^{N+1} D^{(N+1)}(x, \mathbf{y}) \right) \vec{\mathcal{C}} \\ &\quad + x^{N+1} R^{(N+1)}(x, \mathbf{y}) \vec{\mathcal{R}} \\ (\Gamma_\sigma)_* \left(\tilde{Y}_\theta^{(N)} \right) &= Y_0 + \left(d^{(N+1)}(y_1 y_2) + x^{N+1} D_\theta^{(N+1)}(x, \mathbf{y}) \right) \vec{\mathcal{C}} \\ &\quad + x^{N+1} R_\theta^{(N+1)}(x, \mathbf{y}) \vec{\mathcal{R}} \quad . \end{aligned}$$

In fact, we choose:

$$\begin{aligned} D^{(N+1)}(x, \mathbf{y}) &:= \left(\frac{\mathcal{L}_{\tilde{Y}^{(N)}}(\sigma) + x^N \tilde{D}^{(N)}}{x^{N+1}} \right) \circ \Gamma_\sigma^{-1}(x, \mathbf{y}) \\ D_\theta^{(N+1)}(x, \mathbf{y}) &:= \left(\frac{\mathcal{L}_{\tilde{Y}_\theta^{(N)}}(\sigma) + x^N \tilde{D}_\theta^{(N)}}{x^{N+1}} \right) \circ \Gamma_\sigma^{-1}(x, \mathbf{y}) \quad . \end{aligned}$$

By Proposition 2.20, $D^{(N+1)}$ (*resp.* $R^{(N+1)}$) is 1-summable in the direction θ , of 1-sum $D_\theta^{(N+1)}$ (*resp.* $R_\theta^{(N+1)}$). □

3.5. Proof of Proposition 3.1.

We now give a short proof of Proposition 3.1, using the different results proved in this section.

Proof of Proposition 3.1. We just have to use consecutively Proposition 3.6 (applied to $Y_0 := Y|_{\{x=0\}}$), Proposition 3.9, Proposition 3.14 and finally Proposition 3.18, using at each time Corollary 2.21 in order to obtain the directional 1-summability. □

4. SECTORIAL ANALYTIC NORMALIZATION

The aim of this section is to prove that for any $Y \in \mathcal{SN}_{\text{diag},0}$ and for any $\eta \in [\pi, 2\pi[$, there exists a pair

$$(\Phi_+, \Phi_-) \in \text{Diff}_{\text{fib}}(\mathcal{S}_{\arg(i\lambda), \eta}; \text{Id}) \times \text{Diff}_{\text{fib}}(\mathcal{S}_{\arg(-i\lambda), \eta}; \text{Id})$$

whose elements analytically conjugate Y to its normal form Y_{norm} (given by Theorem 1.5) in sectorial neighborhoods of the origin with wide opening. The uniqueness of Φ_+ and Φ_- will be proved in the next section. The existence of sectorial normalizing maps Φ_+ and Φ_- in domains of the form $\mathcal{S}_+ \in \mathcal{S}_{\arg(i\lambda), \eta}$ and $\mathcal{S}_- \in \mathcal{S}_{\arg(-i\lambda), \eta}$ for all $\eta \in [\pi, 2\pi[$, is equivalent to the existence of a sectorial normalizing map Φ_θ in domains $\mathcal{S} \in \mathcal{S}_{\theta, \pi}$, for all $\theta \in \mathbb{R}$ such that $\theta \neq \arg(\pm\lambda)$. In the next section we will also prove that Φ_+ and Φ_- both admit the unique formal normalizing map $\hat{\Phi}$ (given by Theorem 1.5) as weak Gevrey-1 asymptotic expansion in domains $\mathcal{S}_+ \in \mathcal{S}_{\arg(i\lambda), \eta}$ and $\mathcal{S}_- \in \mathcal{S}_{\arg(-i\lambda), \eta}$ respectively. In particular, this will prove that $\hat{\Phi}$ is weakly 1-summable in every direction $\theta \neq \arg(\pm\lambda)$.

We start with a vector field $Y^{(N)}$ normalized up to order $N \geq 2$ as in Proposition 3.1. First of all, we prove the existence of germs of sectorial analytic functions $\alpha_+ \in \mathcal{O}(\mathcal{S}_+)$, $\alpha_- \in \mathcal{O}(\mathcal{S}_-)$, which are solutions to homological equations of the form:

$$\mathcal{L}_{Y^{(N)}}(\alpha_\pm) = x^{M+1} A_\pm(x, \mathbf{y}) \quad ,$$

where $M \in \mathbb{N}_{>0}$ and $A_{\pm} \in \mathcal{O}(\mathcal{S}_{\pm})$ is analytic in \mathcal{S}_{\pm} (see Lemma 4.6). In order to construct such solutions, we will integrate some appropriate meromorphic 1-form on asymptotic paths (see subsection 4.4). Once we have these solutions α_+, α_- , we will construct the desired germs of sectorial diffeomorphisms as the flows of some elementary linear vector fields at “time” $\alpha_{\pm}(x, \mathbf{y})$. After that, we will prove in subsection 5.1 that there exist unique germs of sectorial fibered diffeomorphisms tangent to the identity which conjugate $Y \in \mathcal{SN}_{\text{fib},0}$ to its normal form, by studying the sectorial isotropies in sectorial domains with wide opening.

We go on using the notations introduced in subsection 3.4, *i.e.*

- $\lambda \in \mathbb{C}^*$
- $a_1 + a_2 \notin \mathbb{Q}_{\leq 0}$
- $\vec{\mathcal{C}} := -y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2}$
- $\vec{\mathcal{R}} := y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2}$
- $Y_0 := \lambda \vec{\mathcal{C}} + x \left(x \frac{\partial}{\partial x} + a_1 y_1 \frac{\partial}{\partial y_1} + a_2 y_2 \frac{\partial}{\partial y_2} \right)$.

For $\epsilon \in]0, \frac{\pi}{2}[$ and $r > 0$, we will consider two sectors, namely

$$S_+(r, \epsilon) := S \left(r, \arg(i\lambda) - \frac{\pi}{2} - \epsilon, \arg(i\lambda) + \frac{\pi}{2} + \epsilon \right)$$

and

$$S_-(r, \epsilon) = S \left(r, \arg(-i\lambda) - \frac{\pi}{2} - \epsilon, \arg(-i\lambda) + \frac{\pi}{2} + \epsilon \right).$$

Let us consider a (weakly) 1-summable non-degenerate div-integrable doubly-resonant saddle-node normalized up to an order $N+2$, with $N > 0$:

$$\begin{aligned} Y^{(N+2)} &= Y_0 + \left(c(y_1 y_2) + x^{N+2} D^{(N+2)}(x, \mathbf{y}) \right) \vec{\mathcal{C}} + x^{N+2} R^{(N+2)}(x, \mathbf{y}) \vec{\mathcal{R}} \\ &\quad \text{(formal)} \\ Y_{\pm}^{(N+2)} &= Y_0 + \left(c(y_1 y_2) + x^{N+2} D_{\pm}^{(N+2)}(x, \mathbf{y}) \right) \vec{\mathcal{C}} + x^{N+2} R_{\pm}^{(N+2)}(x, \mathbf{y}) \vec{\mathcal{R}} \\ &\quad \text{(analytic in } S_{\pm}(r, \epsilon) \times \mathbf{D}(\mathbf{0}, \mathbf{r})) \end{aligned}$$

where $D^{(N+2)}, R^{(N+2)}$ are of order at least one with respect to \mathbf{y} , and (weak) 1-summable in every direction $\theta \in \mathbb{R}$ with $\theta \neq \arg(\pm\lambda)$: their respective (weak) 1-sums in the direction $\arg(\pm i\lambda)$ are $D_{\pm}^{(N+2)}, R_{\pm}^{(N+2)}$, which can be analytically extended in $S_{\pm}(r, \epsilon) \times (\mathbb{C}^2, 0)$. In order to have the complete sectorial normalizing map, we have to assume now that our vector field is **strictly non-degenerate**, *i.e.*

$$\boxed{\Re(a_1 + a_2) > 0}.$$

Proposition 4.1. *Under the assumptions above, for all $\eta \in]\pi, 2\pi[$, there exist two germs of sectorial fibered diffeomorphisms*

$$\begin{cases} \Psi_+ \in \text{Diff}_{\text{fib}}(\mathcal{S}_{\arg(i\lambda), \eta}; \text{Id}) \\ \Psi_- \in \text{Diff}_{\text{fib}}(\mathcal{S}_{\arg(-i\lambda), \eta}; \text{Id}) \end{cases}$$

of the form

$$\Psi_{\pm} : (x, \mathbf{y}) \mapsto \left(x, \mathbf{y} + \mathcal{O}(\|\mathbf{y}\|^2) \right),$$

which conjugate $Y_{\pm}^{(N+2)}$ to its formal normal form

$$Y_{\text{norm}} = x^2 \frac{\partial}{\partial x} + (-\lambda + a_1 x - c(y_1 y_2)) y_1 \frac{\partial}{\partial y_1} + (\lambda + a_2 x + c(y_1 y_2)) y_2 \frac{\partial}{\partial y_2},$$

where $c(v) \in v\mathbb{C}\{v\}$ is the germ of an analytic function in $v := y_1 y_2$ vanishing at the origin. Moreover, we can choose Ψ_{\pm} above such that

$$\Psi_{\pm}(x, \mathbf{y}) = \text{Id}(x, \mathbf{y}) + x^N \mathbf{P}_{\pm}^{(N)}(x, \mathbf{y}) ,$$

where $\mathbf{P}_{\pm}^{(N)} = (0, P_{1,\pm}, P_{2,\pm})$ is analytic in $S_{\pm}(r, \epsilon) \times (\mathbb{C}^2, 0)$ (for some $r > 0$ and $\epsilon > \frac{\eta}{2}$) and of order at least two with respect to \mathbf{y} .

By combining Propositions 3.1 and 4.1 we immediately obtain the following result.

Corollary 4.2. *Let $Y \in \mathcal{SN}_{\text{fib},0}$ be a strictly non-degenerate diagonal doubly-resonant saddle-node which is div-integrable. Then, for all $\eta \in]\pi, 2\pi[$, there exist two germs of sectorial fibered diffeomorphisms*

$$\begin{cases} \Phi_+ \in \text{Diff}_{\text{fib}}(\mathcal{S}_{\arg(i\lambda), \eta}; \text{Id}) \\ \Phi_- \in \text{Diff}_{\text{fib}}(\mathcal{S}_{\arg(-i\lambda), \eta}; \text{Id}) \end{cases}$$

tangent to the identity such that:

$$\begin{aligned} (\Phi_{\pm})_*(Y) &= x^2 \frac{\partial}{\partial x} + (-\lambda + a_1 x - c(y_1 y_2)) y_1 \frac{\partial}{\partial y_1} + (\lambda + a_2 x + c(y_1 y_2)) y_2 \frac{\partial}{\partial y_2} \\ &=: Y_{\text{norm}} , \end{aligned}$$

where $\lambda \in \mathbb{C}^*$, $\Re(a_1 + a_2) > 0$, and $c(v) \in v\mathbb{C}\{v\}$ is the germ of an analytic function in $v := y_1 y_2$ vanishing at the origin.

As already mentioned, we prove in the next section that Φ_+ and Φ_- are unique as germs (see Proposition 1.13), and that they are the weak 1-sums of the unique formal normalizing map $\hat{\Phi}$ given by Theorem 1.5 (see Proposition 5.4).

4.1. Proof of Proposition 4.1.

We give here two consecutive propositions which allow to prove Proposition 4.1 as an immediate consequence. When we say that a function $f : U \rightarrow \mathbb{C}$ is *dominated* by another $g : U \rightarrow \mathbb{R}_+$ in U , it means that there exists $L > 0$ such that for all $u \in U$, we have $|f(u)| \leq L.g(u)$.

Proposition 4.3. *Let $Y_{\pm}^{(N+2)} = Y_0 + D_{\pm} \vec{\mathcal{C}} + R_{\pm} \vec{\mathcal{R}}$, where*

$$\begin{cases} D_{\pm}(x, \mathbf{y}) = c(y_1 y_2) + x^{N+2} D_{\pm}^{(N+2)}(x, \mathbf{y}) \\ R_{\pm}(x, \mathbf{y}) = x^{N+2} R_{\pm}^{(N+2)}(x, \mathbf{y}) \end{cases} ,$$

with $N \in \mathbb{N}_{>0}$, $c(v) \in v\mathbb{C}\{v\}$ of order at least one, and $D_{\pm}^{(N+2)}, R_{\pm}^{(N+2)}$ analytic in $S_{\pm}(r, \epsilon) \times (\mathbb{C}^2, 0)$ and dominated by $\|\mathbf{y}\|_{\infty}$. Assume that $\Re(a_1 + a_2) > 0$.

Then, possibly by reducing $r > 0$ and the neighborhood $(\mathbb{C}^2, 0)$, there exist two germs of sectorial fibered diffeomorphisms φ_+ and φ_- in $S_+(r, \epsilon) \times (\mathbb{C}^2, 0)$ and $S_-(r, \epsilon) \times (\mathbb{C}^2, 0)$ respectively, which conjugate $Y_{\pm}^{(N+2)}$ to

$$Y_{\vec{\mathcal{C}}, \pm} := Y_0 + C_{\pm} \vec{\mathcal{C}} ,$$

where $C_{\pm}(x, \mathbf{y}) = D_{\pm} \circ \varphi_{\pm}^{-1}(x, \mathbf{z})$. Moreover we can chose φ_{\pm} to be of the form

$$\varphi_{\pm}(x, \mathbf{y}) = (x, y_1 \exp(\rho_{\pm}(x, \mathbf{y})), y_2 \exp(\rho_{\pm}(x, \mathbf{y}))) ,$$

where $\rho_{\pm}(x, \mathbf{y}) = x^{N+1} \tilde{\rho}_{\pm}(x, \mathbf{y})$ and $\tilde{\rho}_{\pm}$ is analytic in $S_{\pm}(r, \epsilon) \times (\mathbb{C}^2, 0)$ and dominated by $\|\mathbf{y}\|_{\infty}$.

Remark 4.4. Notice that φ_{\pm}^{-1} is of the form

$$\varphi_{\pm}^{-1}(x, \mathbf{z}) = (x, z_1 (1 + x^{N+1} \vartheta(x, \mathbf{z})), z_2 (1 + x^{N+1} \vartheta(x, \mathbf{z}))) ,$$

where ϑ is analytic in $S_{\pm}(r, \epsilon) \times (\mathbb{C}^2, 0)$ and dominated by $\|\mathbf{z}\|_{\infty}$. Consequently:

$$C_{\pm}(x, \mathbf{z}) = c(z_1 z_2) + x^{N+1} C_{\pm}^{(N+1)}(x, \mathbf{z}) ,$$

where c is the same as above and C_{\pm} is analytic in $S_{\pm}(r, \epsilon) \times (\mathbb{C}^2, 0)$ and dominated by $\|\mathbf{z}\|_{\infty}$.

Proposition 4.5. *Let $Y_{C,\pm} := Y_0 + C_\pm \vec{\mathcal{C}}$, where*

$$C_\pm(x, \mathbf{z}) = c(z_1 z_2) + x^{N+1} C_\pm^{(N+1)}(x, \mathbf{z}) \quad ,$$

with $N \in \mathbb{N}_{>0}$, $c(v) \in v\mathbb{C}\{v\}$ of order at least one, and $C_\pm^{(N+1)}$ analytic in $S_\pm(r, \epsilon) \times (\mathbb{C}^2, 0)$ and dominated by $\|\mathbf{z}\|_\infty$. Assume $\Re(a_1 + a_2) > 0$.

Then, possibly by reducing $r > 0$ and the neighborhood $(\mathbb{C}^2, 0)$, there exist two germs of sectorial fibered diffeomorphisms ψ_+ and ψ_- in $S_+(r, \epsilon) \times (\mathbb{C}^2, 0)$ and $S_-(r, \epsilon) \times (\mathbb{C}^2, 0)$ respectively, which conjugate $Y_{C,\pm}$ to

$$Y_{\text{norm}} := Y_0 + c(v) \vec{\mathcal{C}} \quad .$$

Moreover, we can chose ψ_\pm to be of the form

$$\psi_\pm(x, \mathbf{z}) = (x, z_1 \exp(-\chi_\pm(x, \mathbf{z})), z_2 \exp(\chi_\pm(x, \mathbf{z}))) \quad ,$$

where $\chi_\pm(x, \mathbf{z}) = x^N \tilde{\chi}_\pm(x, \mathbf{z})$ and $\tilde{\chi}$ is analytic in $S_\pm(r, \epsilon) \times (\mathbb{C}^2, 0)$ and dominated by $\|\mathbf{z}\|_\infty$.

If we assume for a moment the two propositions above, the proof of Proposition becomes obvious.

Proof of Proposition 4.1. It is an immediate consequence of the consecutive application of the previous two propositions, just by taking $\Psi_\pm = \psi_\pm \circ \varphi_\pm$ with the notations above. \square

4.2. Proof of Propositions 4.3 and 4.5.

In order to prove Propositions 4.3 and 4.5, we will need the following lemmas. The first one gives the existence of analytic solutions (in sectorial domains) to a homological equations we need to solve.

Lemma 4.6. *Let $Z_\pm := Y_0 + C_\pm(x, \mathbf{y}) \vec{\mathcal{C}} + x R_\pm^{(1)}(x, \mathbf{y}) \vec{\mathcal{R}}$, with $C_\pm, R_\pm^{(1)}$ analytic in $S_\pm(r, \epsilon) \times (\mathbb{C}^2, 0)$ and dominated by $\|\mathbf{y}\|_\infty$ and let also $A_\pm(x, \mathbf{y})$ be analytic in $S_\pm(r, \epsilon) \times (\mathbb{C}^2, 0)$ and dominated by $\|\mathbf{y}\|_\infty$. Then for all $M \in \mathbb{N}_{>0}$, possibly by reducing $r > 0$ and the neighborhood $(\mathbb{C}^2, 0)$, there exists a solution α_\pm to the homological equation*

$$(4.1) \quad \mathcal{L}_{Z_\pm}(\alpha_\pm) = x^{M+1} A_\pm(x, \mathbf{y}) \quad ,$$

such that $\alpha_\pm(x, \mathbf{y}) = x^M \tilde{\alpha}_\pm(x, \mathbf{y})$, where $\tilde{\alpha}_\pm$ is a germ of analytic function in $S_\pm(r, \epsilon) \times (\mathbb{C}^2, 0)$ and dominated by $\|\mathbf{y}\|_\infty$.

We will prove this lemma in subsection 4.4. The following lemma proves that φ_\pm and ψ_\pm constructed in Propositions 4.3 and 4.5 are indeed germs of sectorial fibered diffeomorphisms in domains of the form $S_\pm(r, \epsilon) \times (\mathbb{C}^2, 0)$.

Lemma 4.7. *Let f_\pm, g_\pm be two germs of analytic and bounded functions in $S_\pm(r, \epsilon) \times (\mathbb{C}^2, 0)$, which tend to 0 as $(x, \mathbf{y}) \rightarrow (0, \mathbf{0})$ in $S_\pm(r, \epsilon) \times (\mathbb{C}^2, 0)$. Then*

$$\phi_\pm : (x, \mathbf{y}) \mapsto (x, y_1 \exp(f_\pm(x, \mathbf{y})), y_2 \exp(g_\pm(x, \mathbf{y})))$$

defines a germ of sectorial fibered diffeomorphism analytic in $S_\pm(r, \epsilon) \times (\mathbb{C}^2, 0)$ (possibly by reducing $r > 0$ and the neighborhood $(\mathbb{C}^2, 0)$).

Let us explain why these lemmas imply Propositions 4.3 and 4.5.

Proof of both Propositions 4.3 and 4.5. It is sufficient to apply Lemma 4.6 with $M = N + 1$, $A_\pm = -R_\pm^{(N+2)}$, $\alpha_\pm = \rho_\pm$ and $Z_\pm = Y_\pm^{(N+2)}$ for Proposition 4.3, and with $M = N$, $A_\pm = -C_\pm^{(N+1)}$, $\alpha_\pm = \chi_\pm$ and $Z_\pm = Y_{\vec{\mathcal{C}}, \pm}$ for Proposition 4.5. Then we use Lemma 4.7 to see that φ_\pm and ψ_\pm are germs of sectorial fibered diffeomorphisms on the considered domains, and we finally check that they do the

conjugacy we want. With the notations above:

$$\begin{aligned}
 D\varphi_{\pm} \cdot Y_{\pm}^{(N+2)} &= \begin{pmatrix} \mathcal{L}_{Y_{\pm}^{(N+2)}}(x) \\ \mathcal{L}_{Y_{\pm}^{(N+2)}}(y_1 \exp(\rho_{\pm}(x, \mathbf{y}))) \\ \mathcal{L}_{Y_{\pm}^{(N+2)}}(y_2 \exp(\rho_{\pm}(x, \mathbf{y}))) \end{pmatrix} \\
 &= \begin{pmatrix} x^2 \\ \left(\mathcal{L}_{Y_{\pm}^{(N+2)}}(y_1) + y_1 \left(\mathcal{L}_{Y_{\pm}^{(N+2)}}(\rho_{\pm}) \right) \right) \exp(\rho_{\pm}(x, \mathbf{y})) \\ \left(\mathcal{L}_{Y_{\pm}^{(N+2)}}(y_2) + y_2 \left(\mathcal{L}_{Y_{\pm}^{(N+2)}}(\rho_{\pm}) \right) \right) \exp(\rho_{\pm}(x, \mathbf{y})) \end{pmatrix} \\
 &= \begin{pmatrix} x^2 \\ (-\lambda + a_1 x - D_{\pm}(x, \mathbf{y})) y_1 \exp(\rho_{\pm}(x, \mathbf{y})) \\ (\lambda + a_2 x + D_{\pm}(x, \mathbf{y})) y_2 \exp(\rho_{\pm}(x, \mathbf{y})) \end{pmatrix} \\
 &\quad (\text{we have used } \mathcal{L}_{Y_{\pm}^{(N+2)}}(\rho_{\pm}) = -x^{N+2} R_{\pm}^{(N+2)}) \\
 &= (Y_0 + C_{\pm} \vec{\mathcal{C}}) \circ \varphi_{\pm}(x, \mathbf{y}) \\
 &= Y_{\vec{\mathcal{C}}, \pm} \circ \varphi_{\pm}(x, \mathbf{y}),
 \end{aligned}$$

so that $(\varphi_{\pm})_* (Y_{\pm}^{(N+2)}) = Y_{\vec{\mathcal{C}}, \pm}$ and then

$$\begin{aligned}
 D\psi_{\pm} \cdot Y_{\vec{\mathcal{C}}, \pm} &= \begin{pmatrix} \mathcal{L}_{Y_{\vec{\mathcal{C}}, \pm}}(x) \\ \mathcal{L}_{Y_{\vec{\mathcal{C}}, \pm}}(z_1 \exp(-\chi(x, \mathbf{z}))) \\ \mathcal{L}_{Y_{\vec{\mathcal{C}}, \pm}}(z_2 \exp(\chi(x, \mathbf{z}))) \end{pmatrix} \\
 &= \begin{pmatrix} x^2 \\ \left(\mathcal{L}_{Y_{\vec{\mathcal{C}}, \pm}}(z_1) + z_1 \left(\mathcal{L}_{Y_{\vec{\mathcal{C}}, \pm}}(\chi) \right) \right) \exp(-\chi(x, \mathbf{z})) \\ \left(\mathcal{L}_{Y_{\vec{\mathcal{C}}, \pm}}(z_2) + z_2 \left(\mathcal{L}_{Y_{\vec{\mathcal{C}}, \pm}}(\chi) \right) \right) \exp(\chi(x, \mathbf{z})) \end{pmatrix} \\
 &= \begin{pmatrix} x^2 \\ (-\lambda + a_1 x - c(z_1 z_2)) z_1 \exp(-\chi(x, \mathbf{z})) \\ (\lambda + a_2 x + c(z_1 z_2)) z_2 \exp(\chi(x, \mathbf{z})) \end{pmatrix} \\
 &\quad (\text{we have used } \mathcal{L}_{Y_{\vec{\mathcal{C}}, \pm}}(\chi_{\pm}) = -x^{N+1} C_{\pm}^{(N+1)}) \\
 &= (Y_0 + c(u) \vec{\mathcal{C}}) \circ \psi_{\pm}(x, \mathbf{z}) \\
 &= Y_{\text{norm}} \circ \psi_{\pm}(x, \mathbf{z}),
 \end{aligned}$$

so that $(\psi_{\pm})_* (Y_{\vec{\mathcal{C}}, \pm}) = Y_{\text{norm}}$. □

4.3. Proof of Lemma 4.7.

Proof of Lemma 4.7. We consider two germs of analytic functions f_{\pm}, g_{\pm} in $S_{\pm}(r, \epsilon) \times (\mathbb{C}^2, 0)$ which tend to 0 as $(x, \mathbf{y}) \rightarrow (0, \mathbf{0})$ in $S_{\pm}(r, \epsilon) \times (\mathbb{C}^2, 0)$, and we define

$$\phi_{\pm} : (x, \mathbf{y}) \mapsto (x, y_1 \exp(f_{\pm}(x, \mathbf{y})), y_2 \exp(g_{\pm}(x, \mathbf{y}))) .$$

Let us first prove that ϕ_{\pm} is into. Let $\mathbf{x} = (x, y_1, y_2)$ and $\mathbf{x}' = (x', y'_1, y'_2)$ in $S_{\pm}(r, \epsilon) \times (\mathbb{C}^2, 0)$ such that $\phi_{\pm}(\mathbf{x}) = \phi_{\pm}(\mathbf{x}')$. Since ϕ_{\pm} is fibered, necessarily $x = x'$. Then assume that $(y_1, y_2) \neq (y'_1, y'_2)$, such that

$$\|(y_1 - y'_1, y_2 - y'_2)\|_{\infty} > 0$$

and for instance $\|(y_1 - y'_1, y_2 - y'_2)\|_\infty = |y_1 - y'_1| > 0$ (the other case can be done similarly). We denote by $D_{\mathbf{y}}$ the derivative with respect to variables (y_1, y_2) . According to the mean value theorem:

$$\left| \frac{e^{f_\pm(\mathbf{x})} - e^{f_\pm(\mathbf{x}')}}{y_1 - y'_1} \right| \leq \sup_{(z_1, z_2) \in [(y_1, y_2), (y'_1, y'_2)]} \|D_{\mathbf{y}}(e^{f_\pm})(x, z_1, z_2)\|_\infty.$$

Consequently we have:

$$\begin{aligned} 0 &= |y_1 e^{f_\pm(\mathbf{x})} - y'_1 e^{f_\pm(\mathbf{x}')}| \\ &= |e^{f_\pm(\mathbf{x})}| \cdot |y_1 - y'_1| \cdot \left| 1 + \frac{y'_1}{e^{f_\pm(\mathbf{x})}} \cdot \frac{e^{f_\pm(\mathbf{x})} - e^{f_\pm(\mathbf{x}')}}{y_1 - y'_1} \right| \\ &\geq |e^{f_\pm(\mathbf{x})}| \cdot |y_1 - y'_1| \cdot \left(1 - \left| \frac{y'_1}{e^{f_\pm(\mathbf{x})}} \right| \cdot \left| \frac{e^{f_\pm(\mathbf{x})} - e^{f_\pm(\mathbf{x}')}}{y_1 - y'_1} \right| \right) \\ &\geq |e^{f_\pm(\mathbf{x})}| \cdot |y_1 - y'_1| \cdot \left(1 - \left| \frac{y'_1}{e^{f_\pm(\mathbf{x})}} \right| \cdot \sup_{(z_1, z_2) \in [(y_1, y_2), (y'_1, y'_2)]} \|D_{\mathbf{y}}(e^{f_\pm})(x, z_1, z_2)\|_\infty \right) \end{aligned}$$

Assume that we chose $(\mathbb{C}^2, 0) = \mathbf{D}(\mathbf{0}, \mathbf{r})$ small enough such that f_\pm is analytic in

$$S_\pm(r, \epsilon) \times D(0, 3r_1 + \delta) \times D(0, 3r_2 + \delta)$$

with $\delta > 0$ small. Without lost of generality we can take $r_1 = r_2$. We apply Cauchy's integral formula to $z_1 \mapsto e^{f_\pm(x, z_1, z_2)}$, for all fixed z_2 , integrating on the circle of center 0 and radius $3r_1 = 3r_2$. Similarly we also apply Cauchy's integral formula to $z_2 \mapsto e^{f_\pm(x, z_1, z_2)}$, for all fixed z_1 , integrating on the circle of center 0 and radius $3r_2 = 3r_1$. Then we obtain

$$\sup_{(z_1, z_2) \in [(y_1, y_2), (y'_1, y'_2)]} \|D_{\mathbf{y}}(e^{f_\pm})(x, z_1, z_2)\|_\infty \leq \frac{3}{4r_1} \cdot \exp \left(\sup_{\mathbf{x} \in S_\pm(r, \epsilon) \times \mathbf{D}(\mathbf{0}, \mathbf{r})} (|f_\pm(\mathbf{x})|) \right),$$

such that:

$$\begin{aligned} 0 &= |y_1 e^{f_\pm(\mathbf{x})} - y'_1 e^{f_\pm(\mathbf{x}')}| \\ &\geq |e^{f_\pm(\mathbf{x})}| \cdot |y_1 - y'_1| \cdot \left(1 - \frac{3}{4} \exp \left(\sup_{\mathbf{x} \in S_\pm(r, \epsilon) \times \mathbf{D}(\mathbf{0}, \mathbf{r})} (2|f_\pm(\mathbf{x})|) \right) \right). \end{aligned}$$

Since $f_\pm(\mathbf{x}) \xrightarrow{\mathbf{x} \rightarrow \mathbf{0}} 0$, we can choose r, r_1 and r_2 small enough such that:

$$\exp \left(\sup_{\mathbf{x} \in S_\pm(r, \epsilon) \times \mathbf{D}(\mathbf{0}, \mathbf{r})} (2|f_\pm(\mathbf{x})|) \right) \leq \frac{5}{4} < \frac{4}{3}.$$

Finally we obtain:

$$\begin{aligned} 0 &= |y_1 e^{f_\pm(\mathbf{x})} - y'_1 e^{f_\pm(\mathbf{x}')}| \\ &\geq |e^{f_\pm(\mathbf{x})}| \frac{|y_1 - y'_1|}{16} > 0, \end{aligned}$$

and so, if $y_1 \neq y'_1$, $0 = |y_1 e^{f_\pm(\mathbf{x})} - y'_1 e^{f_\pm(\mathbf{x}')}| > 0$, which is a contradiction.

Conclusion: $(y_1, y_2) = (y'_1, y'_2)$ and then ϕ_\pm is into in $S_\pm(r, \epsilon) \times (\mathbb{C}^2, 0)$.

Since ϕ_\pm is into and analytic in $S_\pm(r, \epsilon) \times (\mathbb{C}^2, 0)$, it is a biholomorphism between $S_\pm(r, \epsilon) \times (\mathbb{C}^2, 0)$ and its image which is necessarily open (an analytic function is open), and of the same form. \square

4.4. Resolution of the homological equation: proof of Lemma 4.6.

The goal of this subsection is to prove Lemma 4.6 by studying the existence of paths asymptotic to the singularity and tangent to the foliation, and then to use them to construct the solution to the homological equation (4.1).

For convenience and without loss of generality we assume $\lambda = 1$ during this subsection (otherwise we can divide our vector field by $\lambda \neq 0$, make $x \mapsto \lambda x$ and finally consider $\exp(-i \arg(\lambda)) \cdot S_{\pm}(r, \epsilon)$ instead of $S_{\pm}(r, \epsilon)$: these modifications do not change a_1 and a_2).

4.4.1. Domain of stability and asymptotic paths.

We consider

$$\begin{aligned} Z_{\pm} &= Y_0 + C_{\pm}(x, \mathbf{y}) \vec{\mathcal{C}} + x R_{\pm}^{(1)}(x, \mathbf{y}) \vec{\mathcal{R}} \\ &= \begin{pmatrix} x^2 \\ y_1 \left(-(1 + C_{\pm}(x, \mathbf{y})) + a_1 x + x R_{\pm}^{(1)}(x, \mathbf{y}) \right) \\ y_2 \left(1 + C_{\pm}(x, \mathbf{y}) + a_2 x + x R_{\pm}^{(1)}(x, \mathbf{y}) \right) \end{pmatrix} \end{aligned}$$

with $\Re(a_1 + a_2) > 0$, and $C_{\pm}, R_{\pm}^{(1)}$ analytic in $S_{\pm}(r, \epsilon) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$ and dominated by $\|\mathbf{y}\|_{\infty}$. More precisely, we consider the Cauchy problem of unknown $\mathbf{x}(t) := (x(t), y_1(t), y_2(t))$, with real and increasing time $t \geq 0$, associated to

$$X_{\pm} := \frac{\pm i}{1 + \left(\frac{a_2 - a_1}{2}\right)x + C_{\pm}} Z_{\pm},$$

i.e.

$$(4.2) \quad \begin{cases} \frac{dx}{dt} = \frac{\pm i x^2}{1 + \left(\frac{a_2 - a_1}{2}\right)x + C_{\pm}} \\ \frac{dy_1}{dt} = \frac{\pm i y_1}{1 + \left(\frac{a_2 - a_1}{2}\right)x + C_{\pm}} \left(-(1 + C_{\pm}(x, \mathbf{y})) + a_1 x + x R_{\pm}^{(1)}(x, \mathbf{y}) \right) \\ \frac{dy_2}{dt} = \frac{\pm i y_2}{1 + \left(\frac{a_2 - a_1}{2}\right)x + C_{\pm}} \left(1 + C_{\pm}(x, \mathbf{y}) + a_2 x + x R_{\pm}^{(1)}(x, \mathbf{y}) \right) \\ \mathbf{x}(t) = \mathbf{x}_0 = (x_0, y_{1,0}, y_{2,0}) \in S_{\pm}(r, \epsilon) \times \times \mathbf{D}(\mathbf{0}, \mathbf{r}) \end{cases}.$$

We denote by $(t, \mathbf{x}_0) \mapsto \Phi_{X_{\pm}}^t(\mathbf{x}_0)$ the flow of X_{\pm} with increasing time $t \geq 0$ and with initial point \mathbf{x}_0 : $\Phi_{X_{\pm}}^0(\mathbf{x}_0) = \mathbf{x}_0$.

We will prove the following:

Proposition 4.8. *For all $\epsilon \in \left]0, \frac{\pi}{2}\right[$, there exists finite sectors $S_{\pm}(r, \epsilon), S_{\pm}(r', \epsilon)$ with $r, r' > 0$ and an open domain Ω_{\pm} stable by the flow of (4.2) with increasing time $t \geq 0$ such that*

$$S_{\pm}(r', \epsilon) \times \mathbf{D}(\mathbf{0}, \mathbf{r}') \subset \Omega_{\pm} \subset S_{\pm}(r, \epsilon) \times \mathbf{D}(\mathbf{0}, \mathbf{r}),$$

(cf. figure 4.1). Moreover, if $\mathbf{x}_0 \in \Omega_{\pm}$ then the corresponding solution of (4.2), namely $\mathbf{x}(t) := \Phi_{X_{\pm}}^t(\mathbf{x}_0)$ exists for all $t \geq 0$ and $\mathbf{x}(t) \rightarrow \mathbf{0}$ as $t \rightarrow +\infty$.

Remark 4.9. This will prove that the solution $\mathbf{x}(t)$ to (4.2) exists for all $t \geq 0$ and tends to the origin: it defines a path tangent to the foliation and asymptotic to the origin. Moreover, notice that the domain Ω_{\pm} depends on the choice of r and $r' > 0$.

Definition 4.10. We define the *asymptotic path* with base point $\mathbf{x}_0 \in \Omega_{\pm}$ associated to X_{\pm} the path $\gamma_{\pm, \mathbf{x}_0} := \left\{ \Phi_{X_{\pm}}^t(\mathbf{x}_0), t \geq 0 \right\}$.

For convenience and without loss of generality we only detail the case where “ $\pm = +$ ” (the case where “ $\pm = -$ ” is totally similar).

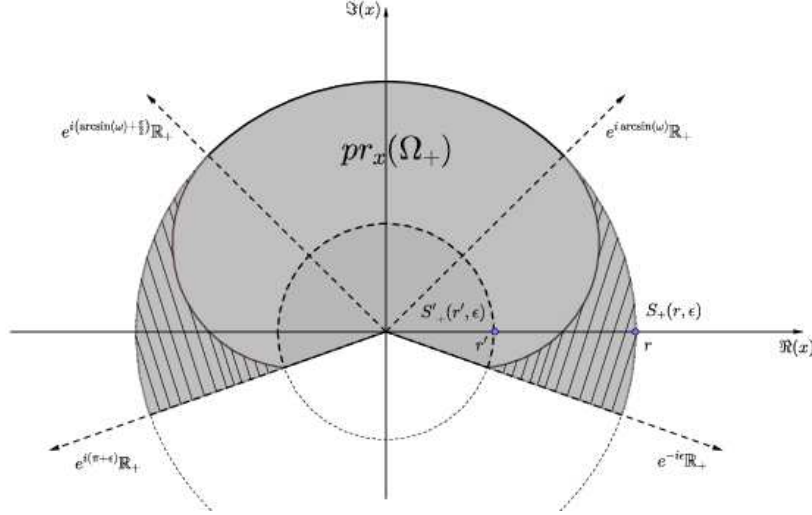


FIGURE 4.1. Representation of the projection $pr_x(\Omega_+)$ of the stable domain Ω_+ in the x -space.

If we write $a := a_1 + a_2$ and $b := \frac{a_2 - a_1}{2}$, in the case “ $\pm = +$ ” we have:

$$\begin{cases} \frac{dx}{dt} = \frac{ix^2}{1+bx+C_+} \\ \frac{dy_1}{dt} = iy_1 \left(-1 + \left(\frac{\frac{a}{2} + R_+^{(1)}(x, \mathbf{y})}{1+bx+C_+(x, \mathbf{y})} \right) x \right) \\ \frac{dy_2}{dt} = iy_2 \left(1 + \left(\frac{\frac{a}{2} + R_+^{(1)}(x, \mathbf{y})}{1+bx+C_+(x, \mathbf{y})} \right) x \right) \\ \mathbf{x}(t) = \mathbf{x}_0 = (x_0, y_{1,0}, y_{2,0}) \in S_+(r, \epsilon) \times \mathbf{D}(\mathbf{0}, \mathbf{r}). \end{cases}$$

We also consider the differential equations satisfied by $|x(t)|$, $|y_1(t)|$, $|y_2(t)|$ and $\theta(t) := \arg(x(t))$:

$$\begin{cases} \frac{d|x(t)|}{dt} = |x(t)| \Re \left(\frac{ix(t)}{1+bx(t)+C_+(\mathbf{x}(t))} \right) \\ \frac{d|y_1(t)|}{dt} = |y_1(t)| \Re \left(ix(t) \left(\frac{\frac{a}{2} + R_+^{(1)}(\mathbf{x}(t))}{1+bx(t)+C_+(\mathbf{x}(t))} \right) \right) \\ \frac{d|y_2(t)|}{dt} = |y_2(t)| \Re \left(ix(t) \left(\frac{\frac{a}{2} + R_+^{(1)}(\mathbf{x}(t))}{1+bx(t)+C_+(\mathbf{x}(t))} \right) \right) \\ \frac{d\theta(t)}{dt} = \Im \left(\frac{ix(t)}{1+bx(t)+C_+(\mathbf{x}(t))} \right). \end{cases}$$

For any non-zero complex number ζ and positive numbers $R, B > 0$, we denote by $\Sigma_+(\zeta, R, B)$ the sector of radius R bisected by $i\bar{\zeta}\mathbb{R}_+$ and of opening $\pi - 2 \arcsin(B) = 2 \arccos(B)$:

$$\begin{aligned} \Sigma_+(\zeta, R, B) &:= \{x \in D(0, R) \mid \Im(\zeta x) > B|\zeta x|\} \\ &= \{x \in D(0, R) \mid -\arccos(B) < \arg(x) - \arg(i\bar{\zeta}) < \arccos(B)\}. \end{aligned}$$

For $T, R > 0$, we denote by $\Theta_+(R, T)$ (*resp.* $\Theta_-(R, T)$) the sector of radius R bisected by \mathbb{R}_+ (*resp.* \mathbb{R}_-) and of opening $2 \arccos(T)$:

$$\begin{aligned}\Theta_+(R, T) &:= \{x \in D(0, R) \mid \Re(x) > T|x|\} \\ &= \{x \in D(0, R) \mid -\arccos(T) < \arg(x) < \arccos(T)\} \\ \Theta_-(R, T) &:= \{x \in D(0, R) \mid \Re(x) < -T|x|\} \\ &= \{x \in D(0, R) \mid -\arccos(T) < \arg(x) - \pi < \arccos(T)\}\end{aligned}$$

Since $\Re(a) > 0$ by assumption, we can choose $\omega' \in]0, \frac{\Re(a)}{|a|}[$, such that $\Sigma_+(a, r, \omega')$ contains $i\mathbb{R}_{>0}$. Indeed, we have

$$|\arg(i) - \arg(i\bar{a})| = |\arg(a)| < \arccos(\omega') .$$

In particular, we have:

$$0 < \arccos(\omega') - |\arg(a)| < \frac{\pi}{2}$$

so that

$$0 < \cos(\arccos(\omega') - |\arg(a)|) < 1 .$$

Hence we take $\omega > 0$ such that

$$(4.3) \quad \omega \in]\cos(\arccos(\omega') - |\arg(a)|), 1[,$$

and then $\Sigma_+(1, r, \omega) \subset \Sigma_+(a, r, \omega')$. Indeed, if $x \in \Sigma_+(1, r, \omega)$, then:

$$(4.4) \quad -\arccos(\omega) < \arg(x) - \frac{\pi}{2} < \arccos(\omega) ,$$

and therefore

$$\begin{aligned}|\arg(x) - \arg(i\bar{a})| &< \arccos(\omega) + |\arg(a)| \\ &\quad (\text{by (4.4)}) \\ &< \arccos(\omega') \\ &\quad (\text{by (4.3)}).\end{aligned}$$

Finally, we fix $\mu \in]0, \sqrt{1 - \omega^2}[$ small enough such that

$$\begin{aligned}\Theta_+(r, \mu) \cap \Sigma_+(1, r, \omega) &\neq \emptyset \\ \Theta_-(r, \mu) \cap \Sigma_+(1, r, \omega) &\neq \emptyset\end{aligned}$$

and

$$S_+(r, \epsilon) \subset \Sigma_+(1, r, \omega) \cup \Theta_+(r, \mu) \cup \Theta_-(r, \mu) .$$

More precisely, we must have $0 < \epsilon < \arccos(\mu)$. The idea is now to study the behavior of $t \mapsto x(t)$ (where $t \mapsto \mathbf{x}(t) = (x(t), y_1(t), y_2(t))$ is the solution of (4.2)) over each domains $\Sigma_+(1, r, \omega)$, $\Theta_+(r, \mu)$, $\Theta_-(r, \mu)$ (*cf.* figure 4.2).

We can now prove the following result, which is a precision of Proposition 4.8.

Lemma 4.11.

- (1) *There exists $r, r_1, r_2 > 0$ such that $\Sigma_+(1, r, \omega) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$ is stable by the flow of (4.2) with increasing time $t \geq 0$. Moreover in this region $|x(t)|$, $|y_1(t)|$ and $|y_2(t)|$ decrease and go to 0 as $t \rightarrow +\infty$.*
- (2) *There exists $0 < r' < r$, $0 < r'_1 < r_1$, $0 < r'_2 < r_2$ and an open domain Ω_+ stable under the action flow of (4.2) with increasing time $t \geq 0$ such that*

$$S_+(r', \epsilon) \times \mathbf{D}(\mathbf{0}, \mathbf{r}') \subset \Omega_+ \subset S_+(r, \epsilon) \times \mathbf{D}(\mathbf{0}, \mathbf{r}) .$$

Moreover, if $x_0 \in \Theta_+(r', \mu)$ (resp. $x_0 \in \Theta_-(r', \mu)$), then $\theta(t) = \arg(x(t))$, $t \geq 0$ is increasing (resp. decreasing) as long as $x(t)$ remains in $\Theta_+(r', \mu)$ (resp. $\Theta_-(r', \mu)$). Finally, there exists $t_0 \geq 0$ such that for all $t \geq t_0$, $x(t) \in \Sigma_+(1, r, \omega)$.

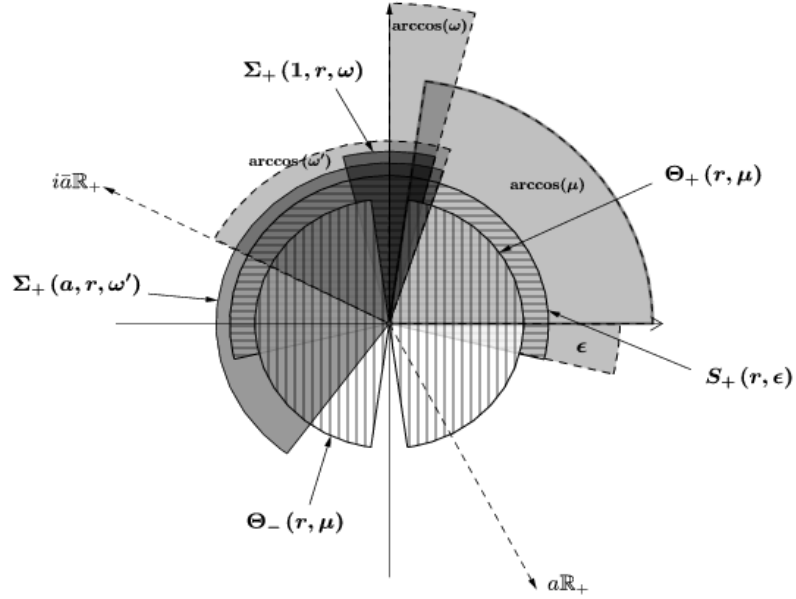


FIGURE 4.2. Representation of domains $\Sigma_+(1, r, \omega)$, $\Sigma_+(a, r, \omega')$, $\Theta_+(r, \mu)$, $\Theta_-(r, \mu)$, $S_+(r, \epsilon)$ (with modified radii for more clarity).

Proof. We fix $\delta \in]0, \min(\omega, \mu)[$, $\delta' \in]0, \omega'[$ and we take $r > 0$ small enough such that for all $\mathbf{x} = (x, \mathbf{y}) \in S_+(r, \epsilon) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$, we have

$$\begin{cases} \left| \frac{1}{1+bx+C_+(\mathbf{x})} - 1 \right| < \delta \\ \left| \frac{\frac{a}{2} + R_+^{(1)}(\mathbf{x})}{1+bx+C_+(\mathbf{x})} - \frac{a}{2} \right| < \delta' \end{cases}.$$

Consequently for all $\mathbf{x} \in S_+ \times \mathbf{D}(\mathbf{0}, \mathbf{r})$ we have the following estimations:

$$\begin{cases} -|x|(1+\delta) < \Re\left(\frac{ix}{1+bx+C_+(\mathbf{x})}\right) < |x|(1+\delta) \\ -|x|(|\frac{a}{2}| + \delta') < \Re\left(ix\left(\frac{\frac{a}{2} + R_+^{(1)}(x, \mathbf{y})}{1+bx+C_+(x, \mathbf{y})}\right)\right) < |x|(|\frac{a}{2}| + \delta') \end{cases}.$$

Moreover:

- if $x \in \Sigma_+(1, r, \omega)$ then

$$\Re\left(\frac{ix}{1+bx+C_+(\mathbf{x})}\right) < -|x|(\omega - \delta) \quad ;$$

- if $x \in \Sigma_+(a, r, \omega')$ (in particular if $x \in \Sigma_+(1, r, \omega)$) then

$$\Re \left(ix \left(\frac{\frac{a}{2} + R_+^{(1)}(x, \mathbf{y})}{1 + bx + C_+(x, \mathbf{y})} \right) \right) < -|x|(\omega' - \delta') \quad ;$$

- if $x \in \Theta_-(r, \mu)$ (*resp.* $\Theta_+(r, \mu)$) then

$$\begin{aligned} \Im \left(\frac{ix}{1 + bx + C_+(\mathbf{x})} \right) &< -|x|(\mu - \delta) \\ \left(\text{resp. } \Im \left(\frac{ix}{1 + bx + C_+(\mathbf{x})} \right) > |x|(\mu - \delta) \right) \end{aligned} \quad .$$

Hence:

- for all $t \geq 0$

$$-(1 + \delta)|x(t)|^2 < \frac{d|x(t)|}{dt} < -(1 + \delta)|x(t)|^2$$

and then, as long as $\mathbf{x}(t) \in S_+(r, \epsilon) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$, we have

$$|x(t)| > \frac{|x_0|}{1 + (1 + \delta)|x_0|t} \quad ;$$

- for all $t \geq 0$, if $x(t) \in \Sigma_+(1, r, \omega)$, then

$$(4.5) \quad \frac{d|x(t)|}{dt} < -(\omega - \delta)|x(t)|^2$$

and

$$(4.6) \quad \begin{cases} \frac{d|y_1(t)|}{dt} < -(\omega' - \delta')|y_1(t)||x(t)| \\ \frac{d|y_2(t)|}{dt} < -(\omega' - \delta')|y_2(t)||x(t)| \end{cases}$$

so that $|x(t)|$, $|y_1(t)|$ and $|y_2(t)|$ are decreasing as long as $x(t) \in \Sigma_+(1, r, \omega)$;

- for all $t \geq 0$, if $x(t) \in \Theta_-(r, \mu)$ (*resp.* $\Theta_+(r, \mu)$) then

$$\begin{aligned} \frac{d\theta}{dt}(t) &< -(\mu - \delta)|x(t)| < \frac{-(\mu - \delta)|x_0|}{1 + (1 + \delta)|x_0|t} \\ \left(\text{resp. } \frac{d\theta}{dt}(t) > (\mu - \delta)|x(t)| > \frac{(\mu - \delta)|x_0|}{1 + (1 + \delta)|x_0|t} \right) \end{aligned}$$

so that $t \mapsto \theta(t)$ is strictly decreasing (*resp.* increasing) as long as $x(t) \in \Theta_-(r, \mu)$ (*resp.* $\Theta_+(r, \mu)$). Moreover, if $\theta_0 = \theta(0)$ is such that $x_0 = x(0) \in \Theta_-(t, \mu) \setminus \Sigma_+(1, r, \omega)$ (*resp.* $\Theta_+(r, \mu) \setminus \Sigma_+(1, r, \omega)$), then as long as $x(t) \in \Theta_-(r, \mu)$ (*resp.* $\Theta_+(r, \mu)$) we have:

$$\begin{aligned} \theta(t) &< \theta_0 - \left(\frac{\mu - \delta}{1 + \delta} \right) \ln(1 + (1 + \delta)|x_0|t) \\ \left(\text{resp. } \theta(t) > \theta_0 + \left(\frac{\mu - \delta}{1 + \delta} \right) \ln(1 + (1 + \delta)|x_0|t) \right) \end{aligned} \quad .$$

We see that $x(t) \in \Sigma_+(1, r, \omega)$ for all

$$t \geq t_0 := \frac{\left(\exp \left(\frac{1+\delta}{\mu-\delta} \left(\theta_0 - \frac{\pi}{2} - \arccos(\omega) \right) \right) - 1 \right)}{(1 + \delta)|x_0|}$$

$$\left(\begin{array}{l} \text{resp. } t_0 := \frac{\left(\exp \left(\frac{1+\delta}{\mu-\delta} \left(\frac{\pi}{2} - \arccos(\omega) - \theta_0 \right) \right) - 1 \right)}{(1+\delta)|x_0|} \end{array} \right) .$$

Indeed, if $t \geq t_0$, with t_0 as above, and if $x(t) \in \Theta_+(r, \mu)$, then we have:

$$\begin{aligned} \theta(t) &> \theta_0 + \left(\frac{\mu - \delta}{1 + \delta} \right) \ln(1 + (1 + \delta)|x_0|t) \\ &> \theta_0 + \left(\frac{\mu - \delta}{1 + \delta} \right) \ln \left(\exp \left(\frac{1 + \delta}{\mu - \delta} \left(\theta_0 - \frac{\pi}{2} - \arccos(\omega) \right) \right) \right) \\ &= \theta_0 + \frac{\pi}{2} - \arccos(\omega) - \theta_0 = \frac{\pi}{2} - \arccos(\omega) \end{aligned}$$

and therefore

$$- \arccos(\omega) < \arg(x(t)) - \frac{\pi}{2} < 0 .$$

Hence, we have $x(t) \in \Sigma_+(1, r, \omega)$. Moreover, notice that

$$(4.7) \quad t_0 \leq \frac{\exp \left(\left(\frac{1+\delta}{\mu-\delta} \right) (\epsilon + \arcsin(\omega)) \right)}{(1+\delta)|x_0|} .$$

On the one hand $\Sigma_+(1, r, \omega) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$ is stable by the flow of (4.2) with increasing time $t \geq 0$. Indeed in this region $|x(t)|, |y_1(t)|$ and $|y_2(t)|$ are decreasing, and as soon as $x(t)$ goes in $\Sigma_+(1, r, \omega) \cap \Theta_-(r, \mu)$ (*resp.* $\Sigma_+(1, r, \omega) \cap \Theta_+(r, \mu)$), which is non-empty and contains a part of the boundary of $\Sigma_+(1, r, \omega)$ with constant argument, $\theta(t)$ is decreasing (*resp.* increasing). Then, $x(t)$ remains in $\Sigma_+(1, r, \omega)$.

On the other hand, as long as $x(t)$ belongs to $\Theta_-(r, \mu)$ (*resp.* $\Theta_+(r, \mu)$) we can reparametrized the solutions by $(-\theta)$ (*resp.* θ) (we are now going to make an abuse of notation, writing when needed $x(\theta)$ or $x(t)$):

$$\left\{ \begin{array}{l} \frac{d|x|}{d(-\theta)} = -|x| \frac{\Re \left(\frac{ix}{1+bx+C_+(\mathbf{x})} \right)}{\Im \left(\frac{ix}{1+bx+C_+(\mathbf{x})} \right)} \leq |x| \cdot \frac{1+\delta}{\mu-\delta} \\ \left(\text{resp. } \frac{d|x|}{d\theta} = |x| \frac{\Re \left(\frac{ix}{1+bx+C_+(\mathbf{x})} \right)}{\Im \left(\frac{ix}{1+bx+C_+(\mathbf{x})} \right)} \leq |x| \cdot \frac{1+\delta}{\mu-\delta} \right) \\ \frac{d|y_1|}{d(-\theta)} = -|y_1| \frac{\Re \left(ix \left(\frac{\frac{a}{2} + R_+^{(1)}(x, \mathbf{y})}{1+bx+C_+(x, \mathbf{y})} \right) \right)}{\Im \left(\frac{ix}{1+bx+C_+(\mathbf{x})} \right)} \leq |y_1| \cdot \frac{|\frac{a}{2}| + \delta'}{\mu-\delta} \\ \left(\text{resp. } \frac{d|y_1|}{d\theta} = |y_1| \frac{\Re \left(ix \left(\frac{\frac{a}{2} + R_+^{(1)}(x, \mathbf{y})}{1+bx+C_+(x, \mathbf{y})} \right) \right)}{\Im \left(\frac{ix}{1+bx+C_+(\mathbf{x})} \right)} \leq |y_1| \cdot \frac{|\frac{a}{2}| + \delta'}{\mu-\delta} \right) \\ \frac{d|y_2|}{d(-\theta)} = -|y_2| \frac{\Re \left(ix \left(\frac{\frac{a}{2} + R_+^{(1)}(x, \mathbf{y})}{1+bx+C_+(x, \mathbf{y})} \right) \right)}{\Im \left(\frac{ix}{1+bx+C_+(\mathbf{x})} \right)} \leq |y_2| \cdot \frac{|\frac{a}{2}| + \delta'}{\mu-\delta} \\ \left(\text{resp. } \frac{d|y_2|}{d\theta} = |y_2| \frac{\Re \left(ix \left(\frac{\frac{a}{2} + R_+^{(1)}(x, \mathbf{y})}{1+bx+C_+(x, \mathbf{y})} \right) \right)}{\Im \left(\frac{ix}{1+bx+C_+(\mathbf{x})} \right)} \leq |y_2| \cdot \frac{|\frac{a}{2}| + \delta'}{\mu-\delta} \right) . \end{array} \right.$$

Hence, if $\theta_0 := \theta(0)$ is such that $x_0 := x(0) \in \Theta_-(r, \mu)$ (*resp.* $\Theta_+(r, \mu)$), for $t \leq t_0$ we have:

$$(4.8) \quad \begin{cases} |x(t)| \leq |x_0| \exp\left(\frac{1+\delta}{\mu-\delta}(\theta_0 - \theta(t))\right) \\ \left(\text{resp. } |x(t)| \leq |x_0| \exp\left(\frac{1+\delta}{\mu-\delta}(\theta(t) - \theta_0)\right)\right) \\ |y_1(t)| \leq |y_{1,0}| \exp\left(\frac{|\frac{\alpha}{2}|+\delta'}{\mu-\delta}(\theta_0 - \theta(t))\right) \\ \left(\text{resp. } |y_1(t)| \leq |y_{1,0}| \exp\left(\frac{|\frac{\alpha}{2}|+\delta'}{\mu-\delta}(\theta(t) - \theta_0)\right)\right) \\ |y_2(t)| \leq |y_{2,0}| \exp\left(\frac{|\frac{\alpha}{2}|+\delta'}{\mu-\delta}(\theta_0 - \theta(t))\right) \\ \left(\text{resp. } |y_2(t)| \leq |y_{2,0}| \exp\left(\frac{|\frac{\alpha}{2}|+\delta'}{\mu-\delta}(\theta(t) - \theta_0)\right)\right). \end{cases}$$

Definition 4.12. We define the domain Ω_+ as the set of all

$$\mathbf{x} = (x, y_1, y_2) \in S_+(r, \epsilon) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$$

such that:

$$\begin{aligned} \bullet \text{ if } \Im(x) \geq \omega |x| \text{ then } & \begin{cases} |x| \leq r \exp\left(\frac{1+\delta}{\mu-\delta}(\arg(x) - \arcsin(\omega))\right) \\ |y_1| \leq r_1 \exp\left(\frac{|\frac{\alpha}{2}|+\delta'}{\mu-\delta}(\arg(x) - \arcsin(\omega))\right) \\ |y_2| \leq r_2 \exp\left(\frac{|\frac{\alpha}{2}|+\delta'}{\mu-\delta}(\arg(x) - \arcsin(\omega))\right) \end{cases} ; \\ \bullet \text{ if } \Im(x) \leq -\omega |x| \text{ then } & \begin{cases} |x| \leq r \exp\left(\frac{1+\delta}{\mu-\delta}(\pi - \arcsin(\omega) - \arg(x))\right) \\ |y_1| \leq r_1 \exp\left(\frac{|\frac{\alpha}{2}|+\delta'}{\mu-\delta}(\pi - \arcsin(\omega) - \arg(x))\right) \\ |y_2| \leq r_2 \exp\left(\frac{|\frac{\alpha}{2}|+\delta'}{\mu-\delta}(\pi - \arcsin(\omega) - \arg(x))\right) \end{cases} . \end{aligned}$$

We see that Ω_+ is stable by the flow of (4.2) with increasing time $t \geq 0$. We have seen that for any initial condition in Ω_+ , the solution exists for any $t \geq 0$, stays in Ω_+ , and after a finite time $t_0 \geq 0$ enters and remains in $\Sigma_+(1, r, \omega)$. Finally, we have:

$$S_+(r', \epsilon) \times \mathbf{D}(\mathbf{0}, \mathbf{r}') \subset \Omega_+ \subset S_+(r, \epsilon) \times \mathbf{D}(\mathbf{0}, \mathbf{r}) ,$$

where

$$\begin{cases} r' = r \exp\left(-\left(\frac{1+\delta}{\mu-\delta}\right)(\epsilon + \arcsin(\omega))\right) < r \\ r'_1 = r_1 \exp\left(-\left(\frac{|\frac{\alpha}{2}|+\delta'}{\mu-\delta}\right)(\epsilon + \arcsin(\omega))\right) < r_1 \\ r'_2 = r_2 \exp\left(-\left(\frac{|\frac{\alpha}{2}|+\delta'}{\mu-\delta}\right)(\epsilon + \arcsin(\omega))\right) < r_2 . \end{cases}$$

Let $\mathbf{x}_0 = (x_0, y_0) \in \Sigma_+(1, r, \omega) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$. From (4.5) and (4.6) we have for all $t \geq 0$:

$$(4.9) \quad \begin{cases} |x(t)| \leq \frac{|x_0|}{1+(\omega-\delta)|x_0|t} \\ |y_1(t)| \leq \frac{|y_{1,0}|}{(1+(1+\delta)|x_0|t)^{\frac{\omega'-\delta'}{1+\delta}}} \\ |y_2(t)| \leq \frac{|y_{2,0}|}{(1+(1+\delta)|x_0|t)^{\frac{\omega'-\delta'}{1+\delta}}} \end{cases} ,$$

which proves that the solutions goes to $\mathbf{0}$ as $t \rightarrow +\infty$. \square

Remark 4.13. Another stable domain Ω_- is defined similarly when dealing with the case “ $\pm = -$ ”

4.4.2. Construction of a sectorial analytic solution to the homological equation.

We consider the meromorphic 1-form $\tau := \frac{dx}{x^2}$, which satisfies $\tau \cdot (Z_{\pm}) = 1$. Let also $A_{\pm}(x, \mathbf{y})$ be analytic in $S_{\pm}(r, \epsilon) \times (\mathbb{C}^2, 0)$ and dominated by $\|\mathbf{y}\|_{\infty}$, and $M \in \mathbb{N}_{>0}$. The following proposition is a precision of Lemma 4.6.

Proposition 4.14. *For all $\mathbf{x}_0 \in \Omega_{\pm}$ (see Definition 4.12), the integral defined by*

$$\alpha_{\pm}(\mathbf{x}_0) := - \int_{\gamma_{\pm, \mathbf{x}_0}} x^{M+1} A_{\pm}(\mathbf{x}) \tau$$

is absolutely convergent (the integration path $\gamma_{\pm, \mathbf{x}_0}$ is the one of Definition 4.10). Moreover, the function $\mathbf{x}_0 \mapsto \alpha_{\pm}(\mathbf{x}_0)$ is analytic in Ω_{\pm} , satisfies

$$\mathcal{L}_{Z_{\pm}}(\alpha_{\pm}) = x^{M+1} A_{\pm}(\mathbf{x})$$

and $\alpha_{\pm}(x, \mathbf{y}) = x^M \tilde{\alpha}_{\pm}(x, \mathbf{y})$, where $\tilde{\alpha}_{\pm}$ is analytic on Ω_{\pm} and dominated by $\|\mathbf{y}\|_{\infty}$.

Proof. We are going to use the estimations obtained in the previous paragraph.

- Let us start by proving that the integral above is convergent. We begin with:

$$\begin{aligned} \alpha_{\pm}(\mathbf{x}_0) &= - \int_0^{+\infty} \frac{x(t)^{M+1} A_{\pm}(\mathbf{x}(t))}{x(t)^2} \frac{ix(t)^2}{1 + bx(t) + C_+(\mathbf{x}(t))} dt \\ &= -i \int_0^{+\infty} \frac{x(t)^{M+1} A_{\pm}(\mathbf{x}(t))}{1 + bx(t) + C_+(\mathbf{x}(t))} dt. \end{aligned}$$

Since $\mathbf{x}(t) \in \Omega_{\pm}$ for all $t \geq 0$ and $A_{\pm}(x, \mathbf{y})$ is dominated by $\|\mathbf{y}\|_{\infty}$, we have then:

$$\left| \frac{x(t)^{M+1} A_{\pm}(\mathbf{x}(t))}{1 + bx(t) + C_+(\mathbf{x}(t))} \right| \leq C |x(t)|^{M+1} \|\mathbf{y}(t)\|_{\infty}$$

where $C > 0$ is some constant, independent of \mathbf{x}_0 and t . For $t \geq 0$ big enough, we deduce from paragraph 4.4.1 that:

$$\begin{aligned} \left| \frac{x(t)^{M+1} A_{\pm}(\mathbf{x}(t))}{1 + bx(t) + C_+(\mathbf{x}(t))} \right| &\leq C \|\mathbf{y}_0\| \left(\frac{|x_0|}{1 + (\omega - \delta)|x_0|t} \right)^{M+1} \frac{1}{(1 + (1 + \delta)|x_0|t)^{\frac{\omega' - \delta'}{1 + \delta}}} \\ &= \underset{t \rightarrow +\infty}{O} \left(\frac{1}{t^{M+1}} \right) \end{aligned}$$

and then the integral is absolutely convergent.

- Let us prove the analyticity of α_{\pm} in Ω_{\pm} : it is sufficient to prove that it is analytic in every compact $K \subset \Omega_{\pm}$. Let K be such a compact subset. Let $L > 0$ such that for all $\mathbf{x} \in K$, we have:

$$\left| \frac{A_{\pm}(\mathbf{x})}{1 + bx + C_+(\mathbf{x})} \right| \leq L.$$

Since K is a compact subset of $\Omega_{\pm} \subset S_{\pm}(r, \epsilon) \times (\mathbb{C}^2, 0)$ and $S_{\pm}(r, \epsilon)$ is open ($0 \notin S_{\pm}(r, \epsilon)$), there exists $\delta > 0$ such that for all $\mathbf{x} = (x, y_1, y_2) \in K$, we have $\delta < |x| < r$. Finally, according to the several estimates in paragraph 4.4.1, there exists $B > 0$ such that for all $\mathbf{x}_0 \in K$ and $t \geq 0$, we have:

$$|x(t)| \leq B \frac{|x_0|}{1 + (\omega - \delta)|x_0|t}.$$

Hence:

$$\begin{aligned} \left| \frac{x(t)^{M+1} A_{\pm}(\mathbf{x}(t))}{1 + bx(t) + C_+(\mathbf{x}(t))} \right| &\leq LB^{M+1} \frac{|x_0|^{M+1}}{(1 + (\omega - \delta)|x_0|t)^{M+1}} \\ &\leq \frac{LB^{M+1} r^{M+1}}{(1 + (\omega - \delta)\delta t)^{M+1}}, \end{aligned}$$

and the classical theorem concerning the analyticity of integral with parameters proves that α_{\pm} is analytic in any compact $K \subset \Omega_{\pm}$, and consequently in Ω_{\pm} .

- Let us write $F(\mathbf{x}) := \frac{\pm ix^{M+1}A_{\pm}(\mathbf{x})}{1+bx+C_{\pm}(\mathbf{x})}$, so that

$$\alpha_{\pm}(\mathbf{x}_0) = - \int_0^{+\infty} F\left(\Phi_{X_{\pm}}^t(\mathbf{x}_0)\right) dt.$$

For all $\mathbf{x}_0 \in \Omega_{\pm}$, the function $t \mapsto \mathbf{x}(t) = \Phi_{X_{\pm}}^t(\mathbf{x}_0)$ satisfies:

$$\frac{\partial}{\partial t} \left(\Phi_{X_{\pm}}^t(\mathbf{x}_0) \right) = \frac{\pm i}{1 + bx \left(\Phi_{X_{\pm}}^t(\mathbf{x}_0) \right) + C_{\pm} \left(\Phi_{X_{\pm}}^t(\mathbf{x}_0) \right)} Z_{\pm} \left(\Phi_{X_{\pm}}^t(\mathbf{x}) \right).$$

The classical theorem about the analyticity of integral with parameters tells us that we can compute the derivatives inside the integral symbol:

$$\begin{aligned} (\mathcal{L}_{Z_{\pm}} \alpha_{\pm})(\mathbf{x}_0) &= - \int_0^{+\infty} \mathcal{L}_{Z_{\pm}}(F \circ \Phi^s)(\mathbf{x}_0) ds \\ &= - \int_0^{+\infty} DF \left(\Phi_{X_{\pm}}^s(\mathbf{x}_0) \right) \cdot D\Phi_{X_{\pm}}^s(\mathbf{x}_0) \cdot Z_{\pm}(\mathbf{x}_0) ds \\ &= - \int_0^{+\infty} DF \left(\Phi_{X_{\pm}}^s(\mathbf{x}) \right) \cdot \frac{\partial}{\partial t} \left(\Phi_{X_{\pm}}^{s+t}(\mathbf{x}_0) \right)_{|t=0} \left(\pm \frac{1 + bx_0 + C_{\pm}(\mathbf{x}_0)}{i} \right) ds \\ &= - \left(\pm \frac{1 + bx_0 + C_{\pm}(\mathbf{x}_0)}{i} \right) \cdot \int_0^{+\infty} DF \left(\Phi_{X_{\pm}}^s(\mathbf{x}_0) \right) \cdot \frac{\partial}{\partial t} \left(\Phi_{X_{\pm}}^t(\mathbf{x}_0) \right)_{|t=s} ds \\ &= - \left(\pm \frac{1 + bx_0 + C_{\pm}(\mathbf{x}_0)}{i} \right) \cdot \int_0^{+\infty} \frac{\partial}{\partial s} \left(F \circ \Phi_{X_{\pm}}^s(\mathbf{x}_0) \right) ds \\ &= - \left(\pm \frac{1 + bx_0 + C_{\pm}(\mathbf{x}_0)}{i} \right) \cdot \left[F \circ \Phi_{X_{\pm}}^s(\mathbf{x}_0) \right]_{s=0}^{s=+\infty} \\ &= - \left(\pm \frac{1 + bx_0 + C_{\pm}(\mathbf{x}_0)}{i} \right) \cdot (-F(\mathbf{x}_0)) \\ &= x_0^{M+1} A_{\pm}(\mathbf{x}_0). \end{aligned}$$

- Let us prove that $\tilde{\alpha}_{\pm}(x, \mathbf{y}) := \frac{\alpha_{\pm}(x, \mathbf{y})}{x^M}$ is bounded and dominated by $\|\mathbf{y}\|_{\infty}$ in Ω_{\pm} . The fact that it is analytic in Ω_{\pm} is clear because α_{\pm} is analytic there and $0 \notin \Omega_{\pm}$. As above, there exists there exists $C > 0$ such that for all $\mathbf{x}_0 := (x_0, \mathbf{y}_0) \in \Omega_{\pm}$ and for all $t \geq 0$:

$$\left| \frac{x \left(\Phi_{X_{\pm}}^t(\mathbf{x}_0) \right)^{M+1} A_{\pm} \left(\Phi_{X_{\pm}}^t(\mathbf{x}_0) \right)}{\left(1 + bx \left(\Phi_{X_{\pm}}^t(\mathbf{x}_0) \right) + C_{\pm} \left(\Phi_{X_{\pm}}^t(\mathbf{x}_0) \right) \right)} \right| \leq C \left| x \left(\Phi_{X_{\pm}}^t(\mathbf{x}_0) \right) \right|^{M+1} \left\| \mathbf{y} \left(\Phi_{X_{\pm}}^t(\mathbf{x}_0) \right) \right\|_{\infty}.$$

We will only deal with the case where $x_0 \in \Theta_{\pm}(r, \mu)$ (the case where $\Sigma_{\pm}(1, r, \omega)$ is easier and can be deduced from that case). On the one hand from (4.8) we have for all $t \leq t_0$:

$$\begin{cases} \left| x \left(\Phi_{X_{\pm}}^t(\mathbf{x}_0) \right) \right| \leq D |x_0| & , \text{ where } D := \exp \left(\frac{1+\delta}{\mu-\delta} (\arccos(\mu) + \epsilon) \right) \\ \left\| \mathbf{y} \left(\Phi_{X_{\pm}}^t(\mathbf{x}_0) \right) \right\|_{\infty} \leq D' \|\mathbf{y}_0\|_{\infty} & , \text{ where } D' := \exp \left(\frac{\left| \frac{\mu}{2} \right| + \delta'}{\mu-\delta} (\arccos(\mu) + \epsilon) \right). \end{cases}$$

On the other hand we have seen in (4.9) that for all $t \geq t_0$:

$$\begin{cases} \left| x \left(\Phi_{X_{\pm}}^t(\mathbf{x}_0) \right) \right| \leq \frac{\left| x \left(\Phi_{X_{\pm}}^{t_0}(\mathbf{x}_0) \right) \right|}{1 + (\omega - \delta) \left| x \left(\Phi_{X_{\pm}}^{t_0}(\mathbf{x}_0) \right) \right|^{(t-t_0)}} \\ \left\| \mathbf{y} \left(\Phi_{X_{\pm}}^t(\mathbf{x}_0) \right) \right\|_{\infty} \leq \|\mathbf{y}_0\|_{\infty} \end{cases}.$$

Hence, we use the Chasles relation and the estimations above to obtain:

$$\begin{aligned}
 |\tilde{\alpha}_{\pm}(x_0, \mathbf{y}_0)| &\leq \frac{|\alpha_{\pm}(x_0, \mathbf{y}_0)|}{|x_0|^M} \\
 &\leq \frac{CD^{M+1}D' \|\mathbf{y}_0\|_{\infty} |x_0|^{M+1} |t_0|}{|x_0|^M} \\
 &\quad + \frac{C \|\mathbf{y}_0\|_{\infty}}{|x_0|^M} \int_{t_0}^{+\infty} \frac{dt}{\left(1 + (\omega - \delta) \left|x \left(\Phi_{X_{\pm}}^t(\mathbf{x}_0)\right)\right| (t - t_0)\right)} \\
 &\leq CD^{M+1}D' \|\mathbf{y}_0\|_{\infty} |x_0| |t_0| + \frac{C \|\mathbf{y}_0\|_{\infty} \left|x \left(\Phi_{X_{\pm}}^{t_0}(\mathbf{x}_0)\right)\right|^{M+1}}{M(\omega - \delta) |x_0|^M \left|x \left(\Phi_{X_{\pm}}^{t_0}(\mathbf{x}_0)\right)\right|} ;
 \end{aligned}$$

and according to (4.7) we have

$$|\tilde{\alpha}_{\pm}(x_0, \mathbf{y}_0)| \leq \left(\frac{D^2 D'}{(1 + \delta)} + \frac{1}{M(\omega - \delta)} \right) CD^M \|\mathbf{y}_0\|_{\infty} .$$

□

5. UNIQUENESS AND WEAK 1-SUMMABILITY

In this section, we prove the uniqueness of the normalizing maps obtained in Corollary 4.2 (see Proposition 1.13) and also the weak 1-summability of the formal normalizing map of Theorem 1.5 (see Proposition 5.4). In particular, we end this section by proving Theorem 1.10.

5.1. Sectorial isotropies in “wide” sectors and uniqueness of the normalizing maps: proof of Proposition 1.13.

We consider a normal form Y_{norm} as given by Corollary 4.2. We study here the germs of sectorial isotropies of the normal form Y_{norm} in $S_{\pm} \times (\mathbb{C}^2, 0)$, where $S_{\pm} \in \mathcal{S}_{\arg(\pm i\lambda), \eta}$ is a sectorial neighborhood of the origin with opening $\eta \in]\pi, 2\pi[$ in the direction $\arg(\pm i\lambda)$. Proposition 1.13 states that the normalizing maps (Φ_+, Φ_-) are unique as sectorial germs. It is a straightforward consequence of Proposition 5.2 below, which show that the only sectorial fibered isotropy (tangent to the identity) of the normal form in over “wide” sector (*i.e.* of opening $> \pi$) is the identity itself.

Definition 5.1. A germ of sectorial fibered diffeomorphism $\Phi_{\theta, \eta}$ in the direction $\theta \in \mathbb{R}$ with opening $\eta \geq 0$ and tangent to the identity, is a germ of fibered sectorial *isotropy* of Y_{norm} (in the direction $\theta \in \mathbb{R}$ with opening $\eta \geq 0$ and tangent to the identity) if $(\Phi_{\theta, \eta})_*(Y_{\text{norm}}) = Y_{\text{norm}}$ in $\mathcal{S} \in \mathcal{S}_{\theta, \eta}$. We denote by $\text{Isot}_{\text{fib}}(Y, \mathcal{S}_{\theta, \eta}; \text{Id}) \subset \text{Diff}_{\text{fib}}(\mathcal{S}_{\theta, \eta}; \text{Id})$ the subset formed composed of these elements.

Proposition 1.13 is an immediate consequence of the following one.

Proposition 5.2. For all $\eta \in]\pi, 2\pi[$:

$$\text{Isot}_{\text{fib}}(Y_{\text{norm}}, \mathcal{S}_{\arg(\pm i\lambda), \eta}; \text{Id}) = \{\text{Id}\} .$$

Proof. Let

$$\phi : (x, \mathbf{y}) \mapsto (x, \phi_1(x, \mathbf{y}), \phi_2(x, \mathbf{y})) \in \text{Isot}_{\text{fib}}(Y_{\text{norm}}, \mathcal{S}_{\arg(\pm i\lambda), \eta}; \text{Id})$$

be a germ of a sectorial fibered isotropy (tangent to the identity) of Y_{norm} in $S_{\pm} \in \mathcal{S}_{\arg(\pm i\lambda), \eta}$ with $\eta \in]\pi, 2\pi[$. Possibly by reducing our domain, we can assume that \mathcal{S}_{\pm} is bounded and of the form $S_{\pm} \times \mathbf{D}(\mathbf{0}, \mathbf{r})$ (where, as usual, S_{\pm} is an adapted sector and $\mathbf{D}(\mathbf{0}, \mathbf{r})$ a polydisc), and that ϕ is bounded in this domain. We have

$$\phi_*(Y_{\text{norm}}) = Y_{\text{norm}}$$

i.e.

$$D\phi \cdot Y_{\text{norm}} = Y_{\text{norm}} \circ \phi$$

which is also equivalent to:

$$(5.1) \quad \begin{cases} x^2 \frac{\partial \phi_1}{\partial x} + (-1 - c(y_1 y_2) + a_1 x) y_1 \frac{\partial \phi_1}{\partial y_1} + (1 + c(y_1 y_2) + a_2 x) y_2 \frac{\partial \phi_1}{\partial y_2} \\ = \phi_1 (-1 - c(\phi_1 \phi_2) + a_1 x) \\ x^2 \frac{\partial \phi_2}{\partial x} + (-1 - c(y_1 y_2) + a_1 x) y_1 \frac{\partial \phi_2}{\partial y_1} + (1 + c(y_1 y_2) + a_2 x) y_2 \frac{\partial \phi_2}{\partial y_2} \\ = \phi_2 (1 + c(\phi_1 \phi_2) + a_2 x) \end{cases}.$$

Let us consider $\psi := \phi_1 \phi_2$. Then

$$x^2 \frac{\partial \psi}{\partial x} + (-1 - c(y_1 y_2) + a_1 x) y_1 \frac{\partial \psi}{\partial y_1} + (1 + c(y_1 y_2) + a_2 x) y_2 \frac{\partial \psi}{\partial y_2} = (a_1 + a_2) x \psi.$$

By assumption we can write

$$\psi(x, \mathbf{y}) = \sum_{j_1 + j_2 \geq 2} \psi_{j_1, j_2}(x) y_1^{j_1} y_2^{j_2},$$

where $\psi_{j_1, j_2}(x)$ is analytic and bounded in S_{\pm} for all $j_1, j_2 \geq 0$ and such that

$$\sum_{j_1 + j_2 \geq 1} \left(\sup_{x \in S_{\pm}} (|\psi_{j_1, j_2}(x)|) \right) y_1^{j_1} y_2^{j_2}$$

is convergent near the origin of \mathbb{C}^2 (e.g. in $\mathbf{D}(\mathbf{0}, \mathbf{r})$). Consequently, with an argument of uniform convergence in every compact subset, we have for all $j_1, j_2 \geq 0$:

$$\begin{aligned} x^2 \frac{d\psi_{j_1, j_2}}{dx}(x) + (j_2 - j_1 + (a_1(j_1 - 1) + a_2(j_2 - 1))x) \psi_{j_1, j_2}(x) \\ = (j_1 - j_2) \sum_{l=1}^{\min(j_1, j_2)} \psi_{j_1-l, j_2-l}(x) c_l. \end{aligned}$$

For $j_1 = j_2 = j \geq 1$, we have

$$\psi_{j, j}(x) = b_{j, j} x^{-(j-1)(a_1 + a_2)}, \quad b_{j, j} \in \mathbb{C}.$$

Since $\Re(a_1 + a_2) > 0$, the function $x \mapsto \psi_{j, j}(x)$ is bounded near the origin if and only if $b_{j, j} = 0$ or $j = 1$. For $j_1 > j_2$, we see recursively that $\psi_{j_1, j_2}(x) = 0$. Indeed, we obtain by induction that

$$\psi_{j_1, j_2}(x) = b_{j_1, j_2} \exp\left(\frac{j_2 - j_1}{x}\right) x^{-(a_1(j_1-1) + a_2(j_2-1))},$$

and since it has to be bounded on S_{\pm} , we necessarily have $b_{j_1, j_2} = 0$. Similarly, for $j_1 < j_2$, we see recursively that $\psi_{j_1, j_2}(x) = 0$. As a conclusion, $\psi(x, \mathbf{y}) = b_{1,1} y_1 y_2 = y_1 y_2$ (we must have $b_{1,1} = 1$ since ϕ is tangent to the identity).

We can now solve separately each equation in (5.1):

$$\begin{cases} x^2 \frac{\partial \phi_1}{\partial x} + (-1 - c(y_1 y_2) + a_1 x) y_1 \frac{\partial \phi_1}{\partial y_1} + (1 + c(y_1 y_2) + a_2 x) y_2 \frac{\partial \phi_1}{\partial y_2} \\ = \phi_1 (-1 - c(y_1 y_2) + a_1 x) \\ x^2 \frac{\partial \phi_2}{\partial x} + (-1 - c(y_1 y_2) + a_1 x) y_1 \frac{\partial \phi_2}{\partial y_1} + (1 + c(y_1 y_2) + a_2 x) y_2 \frac{\partial \phi_2}{\partial y_2} \\ = \phi_2 (1 + c(y_1 y_2) + a_2 x) \end{cases}.$$

As above for $i = 1, 2$ we can write

$$\phi_i(x, \mathbf{y}) = \sum_{j_1 + j_2 \geq 1} \phi_{i, j_1, j_2}(x) y_1^{j_1} y_2^{j_2},$$

where $\phi_{i, j_1, j_2}(x)$ is analytic and bounded in S_{\pm} for all $j_1, j_2 \geq 0$ and such that

$$\sum_{j_1 + j_2 \geq 1} \left(\sup_{x \in S_{\pm}} (|\phi_{i, j_1, j_2}(x)|) \right) y_1^{j_1} y_2^{j_2}$$

is a convergent entire series near the origin of \mathbb{C}^2 (e.g. in $\mathbf{D}(\mathbf{0}, \mathbf{r})$). As above, using the uniform convergence in every compact subset and identifying terms of same homogeneous degree (j_1, j_2) , we obtain:

$$\begin{cases} x^2 \frac{d\phi_{1,j_1;j_2}}{dx}(x) + (j_2 - j_1 + 1 + (a_1(j_1 - 1) + a_2 j_2)x) \phi_{1,j_1;j_2}(x) \\ = \sum_{l=1}^{\min(j_1, j_2)} \phi_{1,j_1-l,j_2-l}(x) (j_1 - j_2 - 1) c_l \\ x^2 \frac{d\phi_{2,j_1;j_2}}{dx}(x) + (j_2 - j_1 - 1 + (a_1 j_1 + a_2(j_2 - 1))x) \phi_{2,j_1;j_2}(x) \\ = \sum_{l=1}^{\min(j_1, j_2)} \phi_{2,j_1-l,j_2-l}(x) (j_1 - j_2 + 1) c_l \end{cases} .$$

From this we deduce:

$$\begin{cases} \phi_{1,1,0}(x) = p_{1,0} \in \mathbb{C} \setminus \{0\} \\ \phi_{2,0,1}(x) = q_{0,1} \in \mathbb{C} \setminus \{0\} \end{cases}$$

with $p_{1,0}q_{0,1} = 1$. Then, using the assumption that $\phi_{i,j_1,j_2}(x)$ is analytic and bounded in S_{\pm} for all $j_1, j_2 \geq 0$, we see (by induction on $j \geq 1$) that

$$\forall j \geq 1 \quad \begin{cases} \phi_{1,j+1,j} = 0 \\ \phi_{2,j,j+1} = 0 \end{cases} .$$

Indeed, we show recursively that for all $j \geq 1$, we have:

$$x^2 \frac{d\phi_{1,j+2,j+1}}{dx}(x) + (j+1)(a_1 + a_2)x \phi_{1,j+2,j+1}(x) = 0 ,$$

and the general solution to this equation is:

$$\phi_{1,j+2,j+1}(x) = p_{j+2,j+1} x^{-(j+1)(a_1+a_2)} , \text{ with } p_{j+2,j+1} \in \mathbb{C} .$$

The quantity $\phi_{1,j+2,j+1}(x)$ is bounded near the origin if and only if $p_{j+2,j+1} = 0$, since $\Re(a_1 + a_2) > 0$. The same arguments work for $\phi_{2,j,j+1}$, $j \geq 1$. Consequently:

$$\begin{cases} x^2 \frac{d\phi_{1,j_1;j_2}}{dx}(x) + (j_2 - j_1 + 1 + (a_1(j_1 - 1) + a_2 j_2)x) \phi_{1,j_1;j_2}(x) \\ = (j_1 - j_2 - 1) \sum_{l=1}^{\min(j_1, j_2)} \phi_{1,j_1-l,j_2-l}(x) c_l \\ x^2 \frac{d\phi_{2,j_1;j_2}}{dx}(x) + (j_2 - j_1 - 1 + (a_1 j_1 + a_2(j_2 - 1))x) \phi_{2,j_1;j_2}(x) \\ = (j_1 - j_2 + 1) \sum_{l=1}^{\min(j_1, j_2)} \phi_{2,j_1-l,j_2-l}(x) c_l \end{cases} .$$

Once again, we see recursively that for $j_1 > j_2 + 1$, $\phi_{1,j_1,j_2}(x) = 0$. Indeed, we obtain by induction that

$$\phi_{1,j_1,j_2}(x) = p_{j_1,j_2} \exp\left(\frac{j_2 - j_1 + 1}{x}\right) x^{-(a_1(j_1-1)+a_2 j_2)} ,$$

and since this has to be bounded on S_{\pm} , we necessarily have $p_{j_1,j_2} = 0$, and therefore $\phi_{1,j_1,j_2}(x) = 0$. Similarly, for $j_1 < j_2 + 1$, we prove that $\phi_{j_1,j_2}(x) = 0$. As a conclusion, $\phi_1(x, \mathbf{y}) = y_1$. By exactly the same kind of arguments we have $\phi_2(x, \mathbf{y}) = y_2$. \square

5.2. Weak 1-summability of the normalizing map.

Let us consider the same data as in Lemma 4.6. The following lemma states that an analytic solution to the considered homological equation in $\mathcal{S}_{\pm} \in \mathcal{S}_{\arg(\pm i\lambda), \eta}$ with $\eta \in [\pi, 2\pi[$, admits a weak Gevrey-1 asymptotic expansion in this sector. In other words, it is the weak 1-sum of a formal solution the homological equation. Let us re-use the notations introduced at the beginning of the latter section 4.

Lemma 5.3. *Let*

$$Z := Y_0 + C(x, \mathbf{y}) \vec{\mathcal{C}} + xR^{(1)}(x, \mathbf{y}) \vec{\mathcal{R}}$$

be a formal vector field weakly 1-summable in $\mathcal{S}_\pm \in \mathcal{S}_{\arg(\pm i\lambda), \eta}$, with $\eta \in [\pi, 2\pi[$ and $C, R^{(1)}$ of order at least one with respect to \mathbf{y} . We denote by

$$Z_\pm := Y_0 + C_\pm(x, \mathbf{y}) \vec{\mathcal{C}} + xR_\pm^{(1)}(x, \mathbf{y}) \vec{\mathcal{R}}$$

the associate weak 1-sum in \mathcal{S}_\pm . Let also $A \in \mathbb{C}[[x, \mathbf{y}]]$ be weakly 1-summable in \mathcal{S}_\pm , of 1-sum A_\pm and of order at least one with respect to \mathbf{y} . Then, any sectorial germ of an analytic function of the form $\alpha_\pm(x, \mathbf{y}) = x^M \tilde{\alpha}_\pm(x, \mathbf{y})$, with $M \in \mathbb{N}_{>0}$ and $\tilde{\alpha}_\pm$ analytic in \mathcal{S}_\pm , which is dominated by $\|\mathbf{y}\|_\infty$ and satisfies

$$\mathcal{L}_{Z_\pm}(\alpha_\pm) = x^{M+1} A_\pm(x, \mathbf{y}) \quad ,$$

has a Gevrey-1 asymptotic expansion in \mathcal{S}_\pm , denoted by α . Moreover, α is a formal solution to

$$\mathcal{L}_Z(\alpha) = x^{M+1} A(x, \mathbf{y}) \quad .$$

Proof. Let us write Z as follow:

$$\begin{aligned} Z = & x^2 \frac{\partial}{\partial x} + (- (\lambda + d(y_1 y_2)) + a_1 x + F_1(x, \mathbf{y})) y_1 \frac{\partial}{\partial y_1} \\ & + (\lambda + d(y_1 y_2) + a_2 x + F_2(x, \mathbf{y})) y_2 \frac{\partial}{\partial y_2} , \end{aligned}$$

with F_1, F_2 weakly 1-summable in $\mathcal{S}_\pm \in \mathcal{S}_{\arg(\pm i\lambda), \eta}$, with $\eta \in [\pi, 2\pi[$, of weak 1-sums $F_{1,\pm}, F_{2,\pm}$ respectively, which are dominated by $\|\mathbf{y}\|$, and with $d(v) \in v\mathbb{C}\{v\}$ without constant term. Consider the Taylor expansion with respect to \mathbf{y} of d, F_1, F_2, A and α :

$$\begin{cases} d(y_1 y_2) = \sum_{k \geq 1} d_k y_1^k y_2^k \\ F_1(x, \mathbf{y}) = \sum_{j_1 + j_2 \geq 1} F_{1,\mathbf{j}}(x) \mathbf{y}^{\mathbf{j}} \\ F_2(x, \mathbf{y}) = \sum_{j_1 + j_2 \geq 1} F_{2,\mathbf{j}}(x) \mathbf{y}^{\mathbf{j}} \\ A(x, \mathbf{y}) = \sum_{j_1 + j_2 \geq 1} A_{\mathbf{j}}(x) \mathbf{y}^{\mathbf{j}} \\ \alpha(x, \mathbf{y}) = \sum_{j_1 + j_2 \geq 1} \alpha_{\mathbf{j}}(x) \mathbf{y}^{\mathbf{j}} \end{cases}$$

(same expansions are valid in \mathcal{S}_\pm for the corresponding weak 1-sums). As usual, possibly by reducing \mathcal{S}_\pm , we can assume that $\mathcal{S}_\pm = \mathcal{S}_\pm \times \mathbf{D}(\mathbf{0}, \mathbf{r})$ (where \mathcal{S}_\pm is an adapted sector and $\mathbf{D}(\mathbf{0}, \mathbf{r})$ a polydisc). The homological equation

$$\mathcal{L}_Z(\alpha) = x^{M+1} A_\pm(x, \mathbf{y})$$

can be re-written:

$$\begin{aligned} x^2 \frac{\partial \alpha}{\partial x} + (- (\lambda + d(y_1 y_2)) + a_1 x + F_{1,\pm}(x, \mathbf{y})) y_1 \frac{\partial \alpha}{\partial y_1} \\ + (\lambda + d(y_1 y_2) + a_2 x + F_{2,\pm}(x, \mathbf{y})) y_2 \frac{\partial \alpha}{\partial y_2} = x^{M+1} A_\pm(x, \mathbf{y}) . \end{aligned}$$

Using normal convergence in any compact subset of \mathcal{S}_\pm , we can compute the partial derivatives of

$$\alpha(x, \mathbf{y}) = \sum_{j_1 + j_2 \geq 1} \alpha_{\mathbf{j}}(x) \mathbf{y}^{\mathbf{j}}$$

with respect to x, y_1 or y_2 term by term, in order to obtain after identification: $\forall \mathbf{j} = (j_1, j_2) \in \mathbb{N}^2$,

$$x^2 \frac{d\alpha_{\mathbf{j},\pm}}{dx}(x) + (\lambda(j_2 - j_1) + (a_1 j_1 + a_2 j_2)x) \alpha_{\mathbf{j},\pm}(x) = G_{\mathbf{j},\pm}(x) \quad ,$$

where $G_{\mathbf{j},\pm}(x)$ depends only on $d_k, F_{1,\mathbf{k},\pm}, F_{2,\mathbf{k},\pm}, \alpha_{\mathbf{k},\pm}$ and $A_{1,\pm}$, for $k \leq \min(j_1, j_2)$, $|\mathbf{k}| \leq |\mathbf{j}| - 1$ and $|\mathbf{l}| \leq |\mathbf{j}|$. We obtain a similar differential equation for the associated formal power series. Let us prove by induction on $|\mathbf{j}| \geq 0$ that:

- (1) $G_{\mathbf{j},\pm}$ is the 1-sum of $G_{\mathbf{j}}$ in S_{\pm} ,
- (2) $G_{j,j}(0) = 0$ if $\mathbf{j} = (j, j)$
- (3) $\alpha_{\mathbf{j},\pm}$ is the 1-sum $\alpha_{\mathbf{j}}$ in S_{\pm} .

It is paramount to use the fact that for all $\mathbf{j} \in \mathbb{N}^2$, $\alpha_{\mathbf{j},\pm}$ is bounded in S_{\pm} .

- For $\mathbf{j} = (0, 0)$, we have $G_{(0,0)} = 0$ and then $\alpha_{(0,0)} = 0$.
- Let $\mathbf{j} = (j_1, j_2) \in \mathbb{N}^2$ with $|\mathbf{j}| = j_1 + j_2 \geq 1$. Assume the property holds for all $\mathbf{k} \in \mathbb{N}^2$ with $|\mathbf{k}| \leq |\mathbf{j}| - 1$.
 - (1) Since $G_{\mathbf{j}}(x)$ depends only on $d_k, F_{1,\mathbf{k}}, F_{2,\mathbf{k}}, \alpha_{\mathbf{k}}$ and A_1 , for $k \leq \min(j_1, j_2)$, $|\mathbf{k}| \leq |\mathbf{j}| - 1$ and $|\mathbf{l}| \leq |\mathbf{j}|$, then $G_{\mathbf{j}}$ is 1-summable in S_{\pm} , of 1-sum $G_{\mathbf{j},\pm}$.
 - (2) We also see that $G_{j,j}(0) = 0$, if $\mathbf{j} = (j, j)$.
 - (3) If $j_1 \neq j_2$, then point 1. in Proposition 2.30 tells us that there exists a unique formal solution $\alpha_{\mathbf{j}}(x)$ to the irregular differential equation we are looking at, and such that $\alpha_{\mathbf{j}}(0) = \frac{1}{\lambda(j_2 - j_1)} G_{\mathbf{j}}(0)$. Moreover, this solution is 1-summable in S_{\pm} since the same goes for $G_{\mathbf{j}}$.
- (4) If however $j_1 = j_2 = j \geq 1$, since $G_{(j,j)}(0) = 0$ we can write $G_{(j,j)}(x) = x \tilde{G}_{(j,j)}(x)$ with $\tilde{G}_{(j,j)}(x)$ 1-summable in S_{\pm} , and then the differential equation becomes regular:

$$x \frac{d\alpha_{(j,j),\pm}}{dx}(x) + (a_1 + a_2) j \alpha_{(j,j),\pm}(x) = \tilde{G}_{(j,j),\pm}(x) .$$

Since $\Re(a_1 + a_2) > 0$, according to point 2. in Proposition 2.30, the latter equation has a unique formal solution $\alpha_{(j,j)}(x)$ such that $\alpha_{(j,j)}(0) = \frac{\tilde{G}_{(j,j)}(0)}{(a_1 + a_2)j}$, and this solution is moreover 1-summable in S_{\pm} , and its 1-sum is the only solution to this equation bounded in S_{\pm} . Thus, it is necessarily $\alpha_{(j,j),\pm}$. □

We are now able to prove the weak 1-summability of the formal normalizing map.

Proposition 5.4. *The sectorial normalizing maps (Φ_+, Φ_-) in Corollary 4.2 are the weak 1-sums in $\mathcal{S}_{\pm} \in \mathcal{S}_{\arg(\pm\lambda), \eta}$ of the formal normalizing map $\hat{\Phi}$ given by Theorem 1.5, for all $\eta \in [\pi, 2\pi[$. In particular, $\hat{\Phi}$ is weakly 1-summable, except for $\arg(\pm\lambda)$.*

Proof. The normalizing map Φ_{\pm} from Corollary 4.2 is constructed as the composition of two germs of sectorial diffeomorphisms, using successively Propositions 3.1 and 4.1. The sectorial map obtained in Proposition 3.1 is 1-summable except in directions $\arg(\pm\lambda)$. The sectorial transformation in Proposition 4.1 is constructed as the composition of two germs of sectorial diffeomorphisms, using successively Proposition 4.3 and 4.5. Both of these two sectorial maps are built thanks to Lemma 4.6. Lemma 5.3 above justifies that each of these maps admits in fact a weak Gevrey-1 asymptotic expansion in a domain of the form $\mathcal{S}_{\pm} \in \mathcal{S}_{\arg(\pm\lambda), \eta}$, for all $\eta \in [\pi, 2\pi[$. Consequently, the same goes for the sectorial diffeomorphisms of Proposition 4.1, and then for those of Corollary 4.2 (we used here Proposition 2.24 for the composition).

Using item 3 in Lemma 2.23, we deduce that the weak Gevrey-1 asymptotic expansion of the sectorial normalizing maps of Corollary 4.2 is therefore a formal normalizing map, such as the one given by Theorem 1.5. By uniqueness of such a normalizing map, it is $\hat{\Phi}$. □

5.3. Proof of Theorem 1.10.

We can now prove Theorem 1.10.

Proof of Theorem 1.10. The existence of Φ_+ and Φ_- is obtained in Corollary 4.2. The uniqueness is given by 1.13. The weak 1-summability is proved in thanks to Proposition 5.4. □

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